Resource Sharing

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Min-max resource sharing

Instance

- finite sets $\mathcal{R}$ of **resources** and $\mathcal{C}$ of **customers**
- for each $c \in \mathcal{C}$:
  - a convex set $\mathcal{B}_c$ of **feasible solutions** (a **block**) and
  - a convex **resource consumption function** $g_c : \mathcal{B}_c \to \mathbb{R}_+^{\mathcal{R}}$
- given by an **oracle function** $f_c : \mathbb{R}_+^{\mathcal{R}} \to \mathcal{B}_c$ with

$$\omega^\top g_c(f_c(\omega)) \leq (1 + \epsilon_0) \inf_{b \in \mathcal{B}_c} \omega^\top g_c(b)$$

for all $\omega \in \mathbb{R}_+^{\mathcal{R}}$ and some $\epsilon_0 \in \mathbb{R}_+$ (a **block solver**).

Task

- Find a $b_c \in \mathcal{B}_c$ for each $c \in \mathcal{C}$ with minimum **congestion**

$$\max_{r \in \mathcal{R}} \sum_{c \in \mathcal{C}} (g_c(b_c))_r \; .$$

Block solvers

A **block solver** is an **oracle function** $f_c : \mathbb{R}_+^{\mathcal{R}} \to \mathcal{B}_c$ with

$$\omega^\top g_c(f_c(\omega)) \leq (1 + \epsilon_0)\, \mathrm{opt}_c(\omega)$$

for all $\omega \in \mathbb{R}_+^{\mathcal{R}}$ and some $\epsilon_0 \in \mathbb{R}_+$, where

$$\mathrm{opt}_c(\omega) := \inf_{b \in \mathcal{B}_c} \omega^\top g_c(b)$$

The block solver is called

- **strong** if $\epsilon_0 = 0$ or $\epsilon_0 > 0$ can be chosen arbitrary small
- **weak** otherwise

The block solver is called

- **bounded** if it can also optimize over

  $$\{b \in \mathcal{B}_c : g_c(b) \leq \mu \mathbb{1}\}$$

  for any given $\mu > 0$ ($c \in \mathcal{C}$).
- **unbounded** otherwise

Width

Let

$$\lambda^* := \inf \left\{ \max_{r\in\mathcal{R}} \sum_{c\in\mathcal{C}} (g_c(b_c))_r : b_c \in \mathcal{B}_c (c \in \mathcal{C}) \right\}$$

(the "**optimum congestion**"), and

$$\rho := \max \left\{ 1, \sup \left\{ \frac{(g_c(b))_r}{\lambda^*} : r \in \mathcal{R}, c \in \mathcal{C}, b \in \mathcal{B}_c \right\} \right\}$$

(the supremum is sometimes called the "**width**" of the problem)

In case of a bounded block solver, and in most applications, we may assume $\rho = 1$ ("no bottleneck").

Summary of results

min-max resource sharing	block solver	running time
Grigoriadis, Khachiyan [1994]	strong, bounded	$\tilde{O}(\epsilon^{-2}\|\mathcal{C}\|^2\theta)$
Grigoriadis, Khachiyan [1996]	strong, unbounded	$\tilde{O}(\epsilon^{-2}\|\mathcal{C}\|\|\mathcal{R}\|\theta)$
Jansen, Zhang [2008]	weak, unbounded	$\tilde{O}(\epsilon^{-2}\|\mathcal{C}\|\|\mathcal{R}\|\theta)$
Müller, V. [2008]	weak, unbounded	$\tilde{O}(\epsilon^{-2}\rho\|\mathcal{C}\|\theta)$
Müller, V. [2008]	weak, bounded	$\tilde{O}(\epsilon^{-2}\|\mathcal{C}\|\theta)$

fractional packing (all g_c linear)	block solver	running time
Plotkin, Shmoys, Tardos [1995] $*$	strong, unbounded	$\tilde{O}(\epsilon^{-2}\rho\|\mathcal{C}\|\theta)$
Young [1995]	weak, unbounded	$\tilde{O}(\epsilon^{-2}\rho\|\mathcal{C}\|\theta)$
Charikar et al. [1998] $*$	weak, unbounded	$\tilde{O}(\epsilon^{-2}\rho\|\mathcal{C}\|\theta)$
Bienstock, Iyengar [2004]	—	$\tilde{O}(\epsilon^{-1}\cdots)$

Algorithms compute a $(1+\epsilon_0+\epsilon)$-approximate solution.
Running times for fixed $\epsilon_0 \geq 0$. Logarithmic terms omitted.
Entries with $*$ refer to the feasibility version ($\lambda^* = 1$).

Weak duality

Lemma (Weak duality)

Let $\omega \in \mathbb{R}_+^{\mathcal{R}}$ be some cost vector with $\omega^\top \mathbb{1} \neq 0$. Then

$$\frac{\sum_{c \in \mathcal{C}} opt_c(\omega)}{\omega^\top \mathbb{1}} \leq \lambda^*.$$

Proof

Let $(b_c \in \mathcal{B}_c)_{c \in \mathcal{C}}$ be a solution with congestion λ^*. Then

$$\frac{\sum_{c \in \mathcal{C}} \mathrm{opt}_c(\omega)}{\omega^\top \mathbb{1}} \leq \frac{\sum_{c \in \mathcal{C}} \omega^\top g_c(b_c)}{\omega^\top \mathbb{1}} = \frac{\omega^\top \sum_{c \in \mathcal{C}} g_c(b_c)}{\omega^\top \mathbb{1}} \leq \frac{\omega^\top \lambda^* \mathbb{1}}{\omega^\top \mathbb{1}} = \lambda^*$$

$\square$

Bounding λ^*

Lemma (Weak duality)

Let $\omega \in \mathbb{R}_+^{\mathcal{R}}$ *be some cost vector with* $\omega^\top \mathbb{1} \neq 0$. *Then*

$$\frac{\sum_{c \in \mathcal{C}} opt_c(\omega)}{\omega^\top \mathbb{1}} \leq \lambda^*.$$

Corollary

Let $b_c := f_c(\mathbb{1})$ *(*$c \in \mathcal{C}$*) and* $\lambda^{ub} := \max_{r \in \mathcal{R}} \sum_{c \in \mathcal{C}} (g_c(b_c))_r$. *Then*

$$\frac{\lambda^{ub}}{|\mathcal{R}|(1+\epsilon_0)} \leq \frac{\sum_{r \in \mathcal{R}} \sum_{c \in \mathcal{C}} (g_c(b_c))_r}{|\mathcal{R}|(1+\epsilon_0)} \leq \frac{\sum_{c \in \mathcal{C}} opt_c(\mathbb{1})}{|\mathcal{R}|} \leq \lambda^* \leq \lambda^{ub}\,.$$

$\square$

Scaling and binary search

We know $\frac{\lambda^{ub}}{|\mathcal{R}|(1+\epsilon_0)} \leq \lambda^* \leq \lambda^{ub}$.

1. Set $j := 0$.
2. Scale $g_c^{(j)}(b) := g_c(b)\frac{2^j}{\lambda^{ub}}$. Note that $\lambda^{*(j)} \leq 1$.
3. Find a solution with congestion $\lambda^{(j)} \leq (1 + \epsilon_0 + \frac{1}{4})\lambda^{*(j)} + \frac{1}{4}$.
4. If $\lambda^{(j)} \leq \frac{1}{2}$, then increment j and go to 2.
5. Now $\frac{1}{5(1+\epsilon_0)} \leq \lambda^{*(j)} \leq 1$.
6. Find a solution with congestion $\lambda^{(j)} \leq (1 + \epsilon_0 + \frac{\epsilon}{6})\lambda^{*(j)} + \frac{\epsilon}{6(1+\epsilon)}$.

Lemma (Main Lemma)

Let $\delta, \delta' > 0$. Suppose that $\lambda^* \leq 1$.
Then we can compute a solution with congestion at most

$$(1 + \epsilon_0 + \delta)\lambda^* + \delta'$$

in

$$O\left((\delta\delta')^{-1}|\mathcal{C}|\theta\rho(1 + \epsilon_0)^2 \log |\mathcal{R}|\right)$$

time, where θ is the time for an oracle call.

Core algorithm

Input: An instance of the min-max resource sharing problem.
Output: A convex combination of vectors in $\mathcal{B}_c$ for each $c \in \mathcal{C}$.

Set $t := \left\lceil \frac{4\rho(1+\epsilon_0)^2 \ln |\mathcal{R}|}{\delta' \min\{1,\delta\}} \right\rceil$.

Set $\alpha_r := 0$ and $\omega_r := 1$ for each $r \in \mathcal{R}$.
Set $x_{c,b} := 0$ for each $c \in \mathcal{C}$ and $b \in \mathcal{B}_c$.
For $p := 1$ to t **do**: (perform t **phases**)
    For each $c \in \mathcal{C}$ **do**:
        AllocateResources(c).
Set $x_{c,b} := \frac{1}{t} x_{c,b}$ for each $c \in \mathcal{C}$ and $b \in \mathcal{B}_c$. (normalize)

Core algorithm: subroutine

Set $\epsilon_2 := \frac{\min\{1,\delta\}}{4\rho(1+\epsilon_0)^2}$.

Procedure AllocateResources(c):

Set $b_c := f_c(\omega)$. (call oracle)

Set $x_{c,b_c} := x_{c,b_c} + 1$.

Set $\alpha := \alpha + g_c(b_c)$. (update resource consumption)

For each $r \in \mathcal{R}$ with $(g_c(b_c))_r \neq 0$ **do**:

Set $\omega_r := e^{\epsilon_2 \alpha_r}$. (update prices)

Proof of performance guarantee (sketch)

Lemma

Let (x, ω) be the output of the algorithm, and let

$$\lambda_r := \sum_{c \in \mathcal{C}} \left(g_c \left(\sum_{b \in \mathcal{B}_c} x_{c,b} b \right) \right)_r$$

and $\lambda := \max_{r \in \mathcal{R}} \lambda_r$. Then

$$\lambda \leq \frac{1}{\epsilon_2 t} \ln \sum_{r \in \mathcal{R}} e^{\epsilon_2 t \lambda_r} = \frac{1}{\epsilon_2 t} \ln\left(\omega^\top \mathbb{1}\right).$$

Proof: Since the functions g_c are convex, we have for $r \in \mathcal{R}$:

$$\lambda_r \leq \sum_{c \in \mathcal{C}} \sum_{b \in \mathcal{B}_c} x_{c,b} (g_c(b))_r = \frac{\alpha_r}{t} = \frac{1}{\epsilon_2 t} \ln\left(e^{\epsilon_2 \alpha_r}\right) = \frac{1}{\epsilon_2 t} \ln \omega_r$$

$\square$

Proof of performance guarantee (sketch)

Lemma (Main Lemma)

Let $\delta, \delta' > 0$. Suppose that $\lambda^* \leq 1$.
Then the algorithm computes a solution with congestion at most

$$(1 + \epsilon_0 + \delta)\lambda^* + \delta' \,.$$

Sketch of proof:

- Congestion is at most $\frac{1}{\epsilon_2 t} \ln\big((\omega^{(t)})^\top \mathbb{1}\big)$.
- Initially, we have $(\omega^{(0)})^\top \mathbb{1} = |\mathcal{R}|$.
- Short calculation yields

$$(\omega^{(p)})^\top \mathbb{1} \;\leq\; (\omega^{(p-1)})^\top \mathbb{1} \;+\; \epsilon' \sum_{c \in \mathcal{C}} \mathrm{opt}_c(\omega^{(p)}),$$

where $\omega^{(i)}$ is the price vector at the end of the *i*-th phase and $\epsilon' := \epsilon_2(1 + (e-2)\rho\epsilon_2)(1 + \epsilon_0)$.

Proof of performance guarantee (sketch)

We had $(\omega^{(p)})^\top \mathbb{1} \leq (\omega^{(p-1)})^\top \mathbb{1} + \epsilon' \sum_{c \in \mathcal{C}} \mathrm{opt}_c(\omega^{(p)})$.

By weak duality, $\epsilon' \frac{\sum_{c \in \mathcal{C}} \mathrm{opt}_c(\omega^{(p)})}{(\omega^{(p)})^\top \mathbb{1}} \leq \epsilon' \lambda^* < 1$, and we get

$$(\omega^{(p)})^\top \mathbb{1} \leq \frac{1}{1 - \epsilon' \lambda^*} (\omega^{(p-1)})^\top \mathbb{1}$$

and thus

$$(\omega^{(t)})^\top \mathbb{1} \leq \frac{|\mathcal{R}|}{(1 - \epsilon' \lambda^*)^t} = |\mathcal{R}| \left(1 + \frac{\epsilon' \lambda^*}{1 - \epsilon' \lambda^*}\right)^t \leq |\mathcal{R}| e^{t \epsilon' \lambda^* / (1 - \epsilon' \lambda^*)} .$$

Together with $\lambda \leq \frac{1}{\epsilon_2 t} \ln\big((\omega^{(t)})^\top \mathbb{1}\big)$, this proves the claim. $\square$

Main result

Theorem

The presented algorithm computes a $(1+\epsilon_0+\epsilon)$*-approximate solution in* $O(|\mathcal{C}|\theta\rho(1+\epsilon_0)^2 \log|\mathcal{R}|(\log|\mathcal{R}|+\epsilon^{-2}(1+\epsilon_0)))$ *time, where* θ *is the time for an oracle call.*

(Müller, V. [2008])

Extensions for practical application:

- Most oracle calls not necessary; reuse previous result if still good enough. Use lower bounds to decide
- Speed-up heuristics
- Randomized rounding to extreme points of the blocks
- Re-choose where rounding violates constraints

Application to global routing

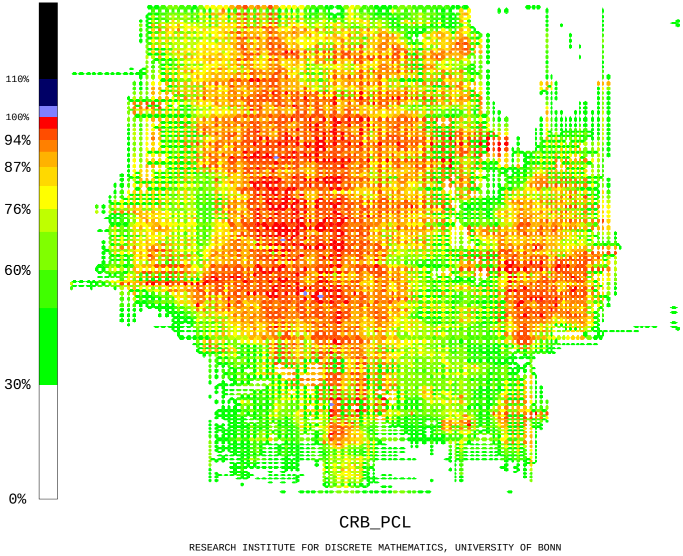
Given a global routing graph (3D grid with millions of vertices).

- **Customers** = nets (sets of pins; roughly: sets of vertices)
- **Resources** = edge capacities, power consumption, yield loss, timing constraints, ...
- Objective function is transformed into a constraint
- **Block** = (convex hull of) set of Steiner trees for a net, with space consumption for each edge
- Resource consumption is nonlinear (but convex) for yield loss, timing, power consumption
- **Block solver** = approximation algorithm for the Steiner tree problem in the global routing graph (with edge weights)

The algorithm in practice

- In practice, results are much better than theoretical performance guarantees. Usually 10–20 iterations suffice.
- Only few upper bounds are violated; these are corrected easily by *rip-up and re-route*.
- Detailed routing can realize the solution well, due to excellent capacity estimations.
- Small integrality gap and approximate dual solution implies that an infeasibility proof can be found for most infeasible instances.

Congestion map of a difficult instance



Running time in practice

Chip	$\lvert\mathcal{C}\rvert$	$\lvert\mathcal{R}\rvert$	1 thread	4 threads	8 threads
A	478,946	894,377	0:15:49	0:04:25	0:02:37
B	786,368	1,949,245	1:18:13	0:23:09	0:14:29
C	529,966	1,091,339	0:48:40	0:13:19	0:08:20
D	959,163	2,794,166	1:12:26	0:21:00	0:10:49
E	3,590,647	20,392,657	1:16:07	0:23:27	0:15:09
F	5,340,123	23,606,915	0:33:25	0:12:22	0:08:51
G	7,039,094	22,891,145	2:32:48	0:46:12	0:29:08

Summary

- Min-max resource sharing is a very general problem
- We can solve it efficiently for millions of customers and resources
- Yields provably near-optimum solutions for global routing
- Core global optimization of overall routing flow

Thank you!