Shorter Tours by Nicer Ears

7/5-approximation for graphic TSP, 3/2 for the path version, and 4/3 for two-edge-connected subgraphs

Jens Vygen

(joint work with András Sebő)

September 21, 2012

Metric TSP

Given a complete graph G and metric weights $c : E(G) \to \mathbb{R}_{\geq 0}$, find a Hamiltonian circuit in G with minimum total weight.

- *NP*-hard
- best known approximation ratio $\frac{3}{2}$ (Christofides [1976])
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We will show an approximation ratio of 1.4.

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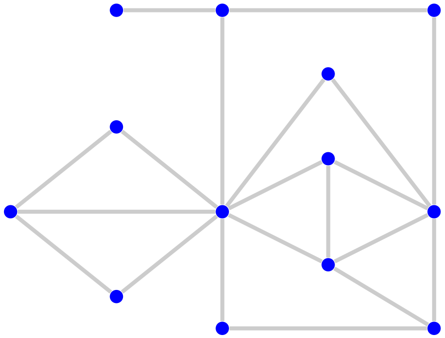

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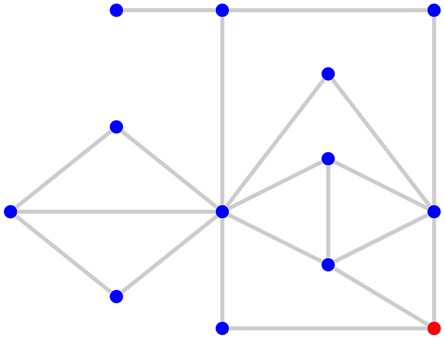

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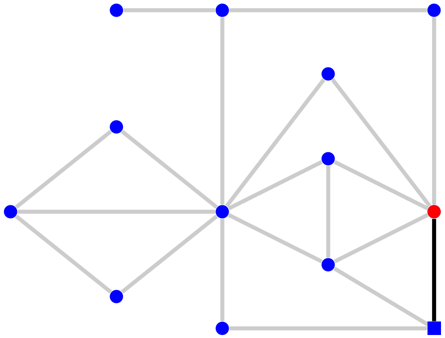

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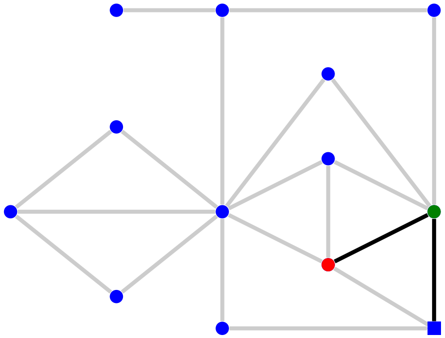

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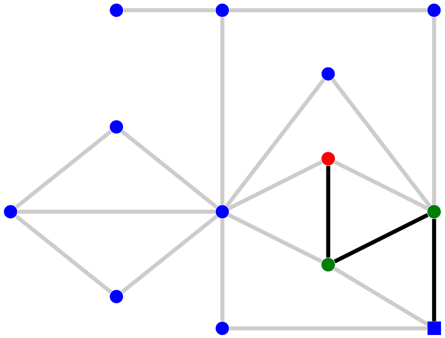

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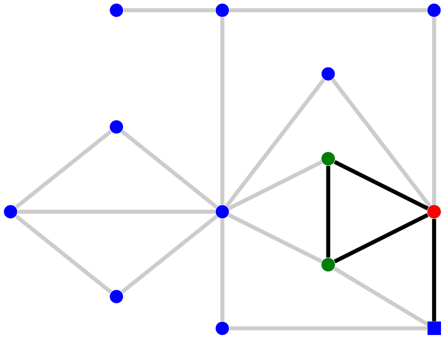

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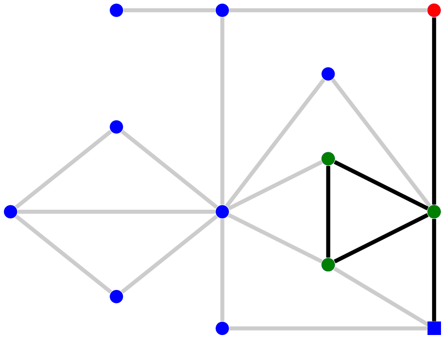

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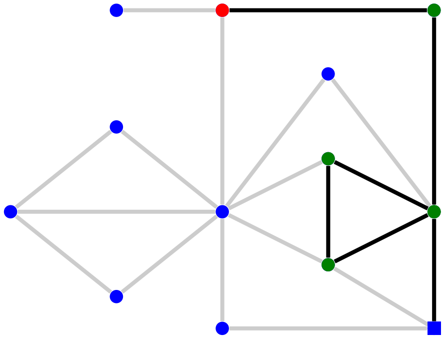

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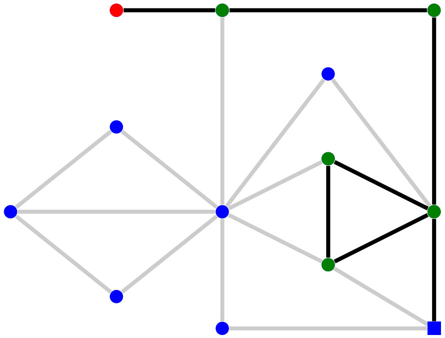

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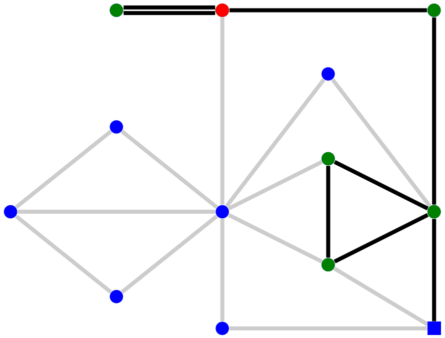

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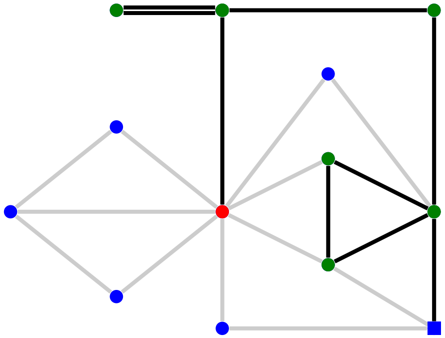

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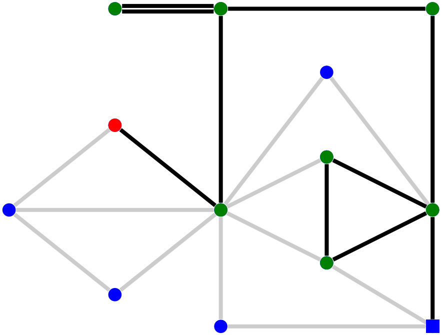

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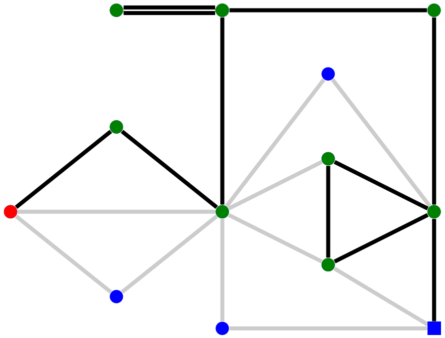

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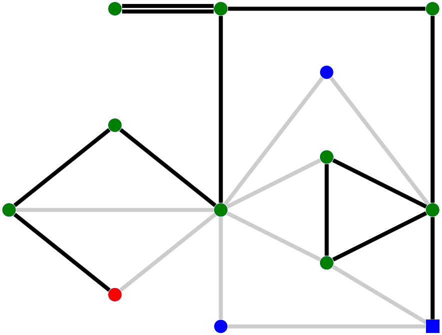

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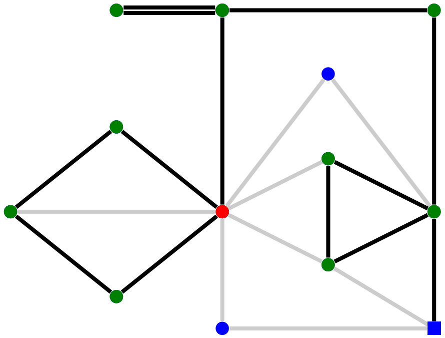

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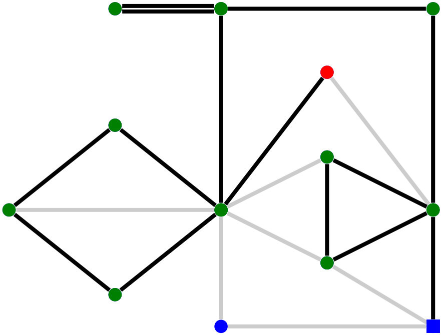

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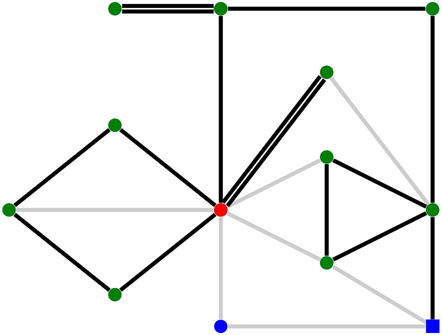

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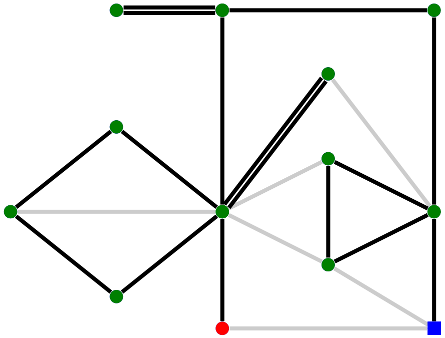

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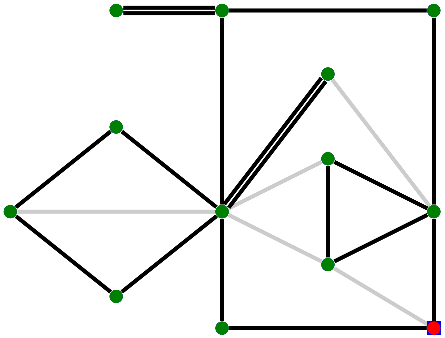

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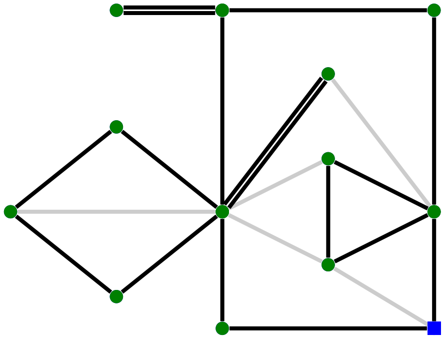

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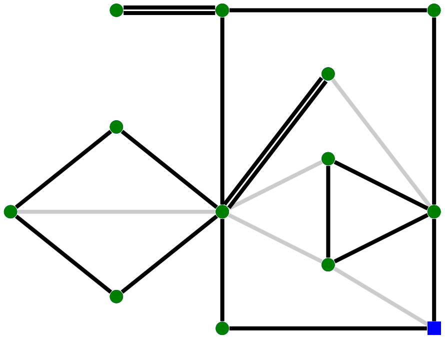

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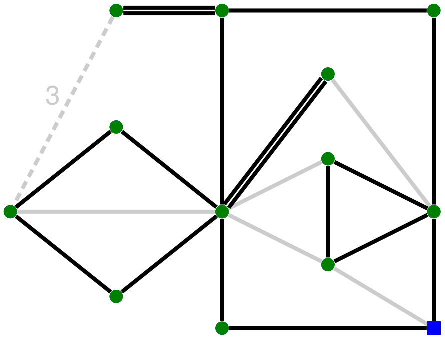

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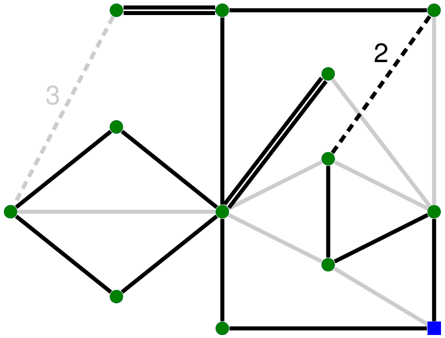

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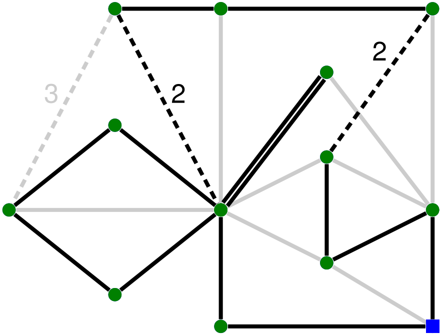

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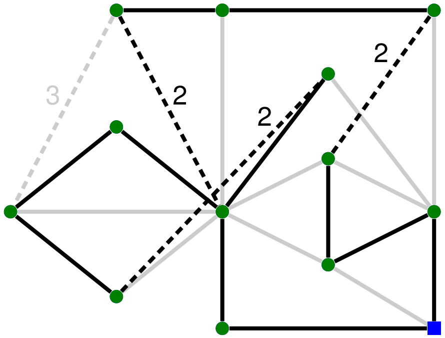

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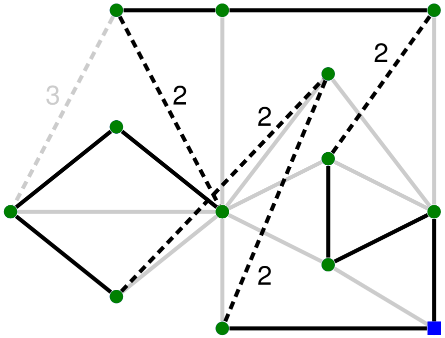

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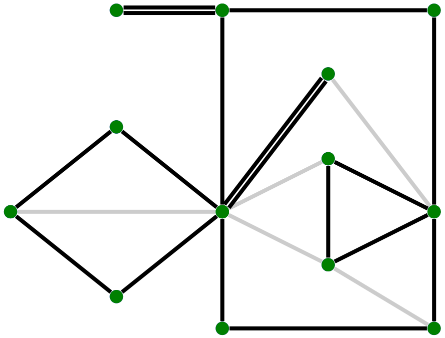

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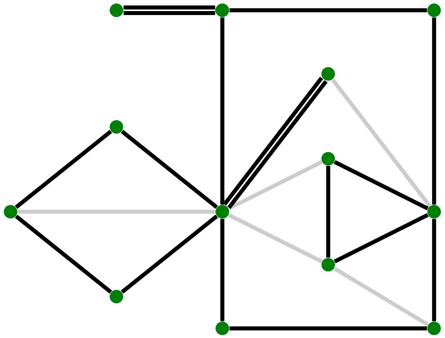

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- find a smallest $\underbrace{\text{Eulerian spanning subgraph of } 2G}_{\text{its edge set is called tour or connected-}\emptyset\text{-join of } G}$

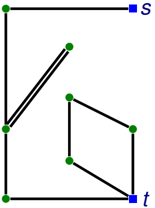
Main results

Let G be a connected graph and $T \subseteq V(G)$ with $|T|$ even.

A connected-T-join of G (aka T-tour) is a set $F \subseteq E(2G)$ such that

- $(V(G), F)$ is connected, and
- $v \in T \Leftrightarrow |F \cap \delta(v)|$ odd.

$T = \{s, t\}$



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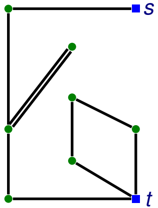

We improve the approximation ratio for:

- Graphic TSP (smallest tour = connected-$\emptyset$-join): from $\frac{13}{9}$ (Mucha [2012]) to $\frac{7}{5}$
- Connected-T-join (smallest connected-T-join): from $\frac{5}{3}$ (Christofides [1976], Hoogeveen [1991]), 1.578 for $T = \{s, t\}$ (An, Kleinberg, Shmoys [2012]), to $\frac{3}{2}$
- 2ECSS (smallest 2-edge-connected spanning subgraph): from $\frac{17}{12}$ (Cheriyan, Sebő, Szigeti [2001]) to $\frac{4}{3}$

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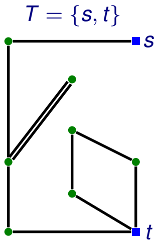

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Note that doubling edges is necessary (except for 2ECSS) and tripling edges does not help.

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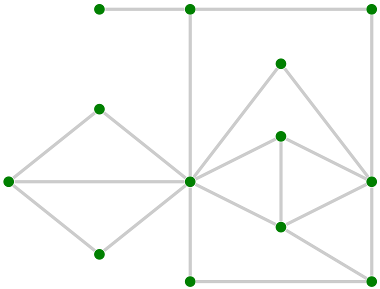

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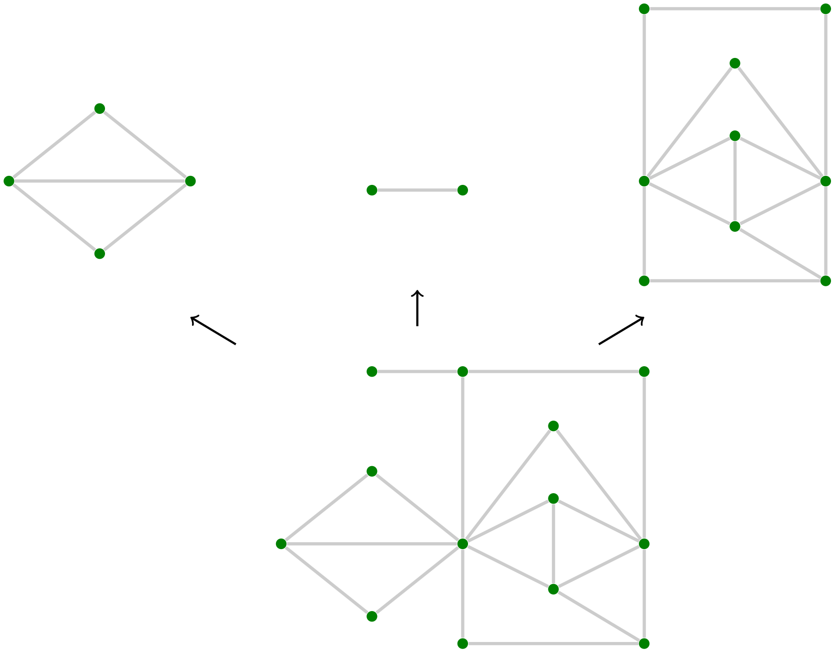

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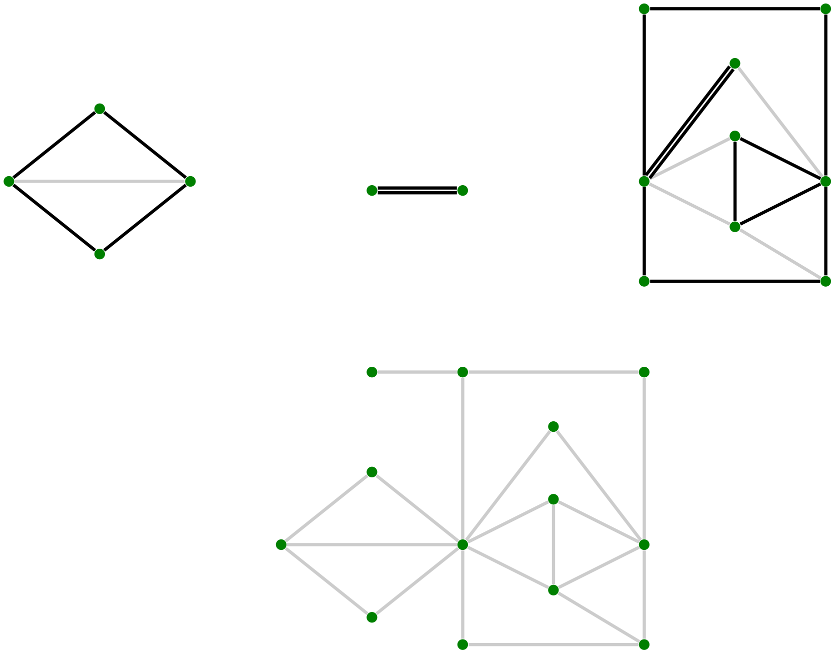

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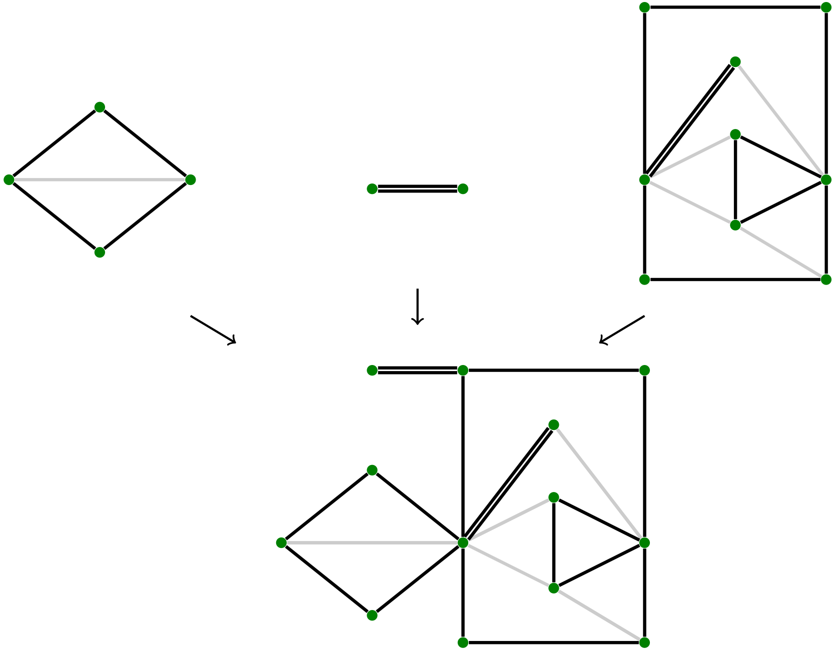

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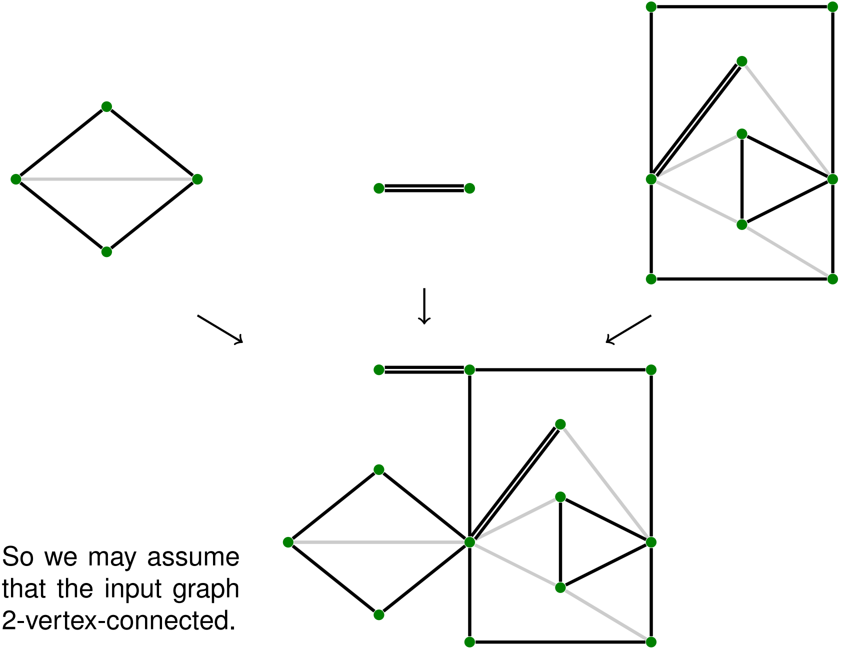

So we may assume that the input graph 2-vertex-connected.

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Write $G = P_0 + P_1 + \cdots + P_k$, where P_0 is a single vertex, and each P_i ($i = 1, \ldots, k$) is either

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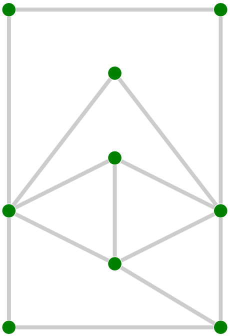

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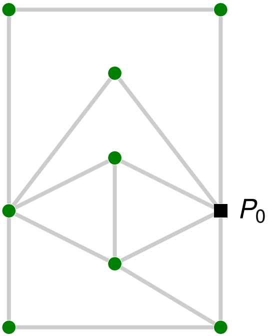

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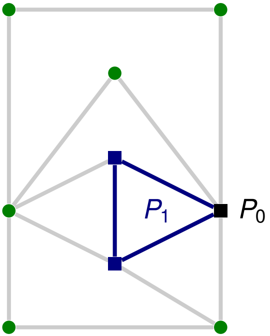

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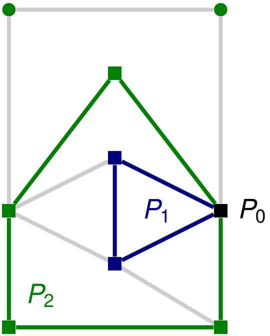

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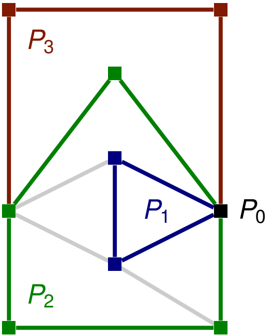

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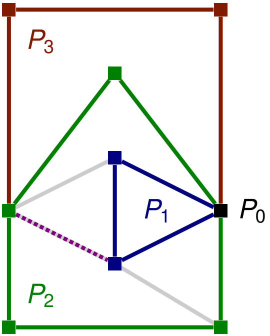

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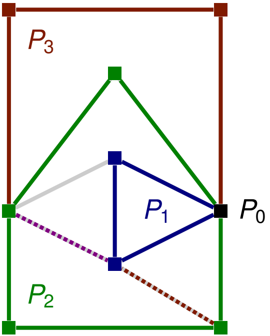

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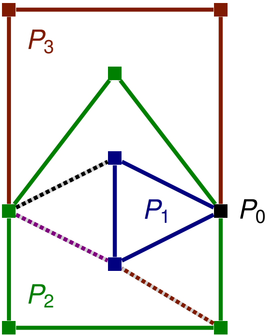

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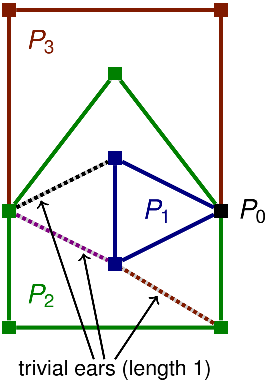

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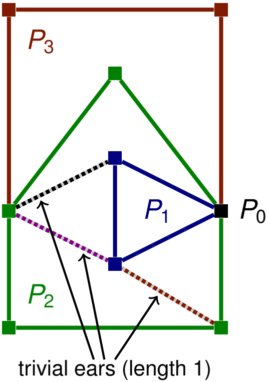

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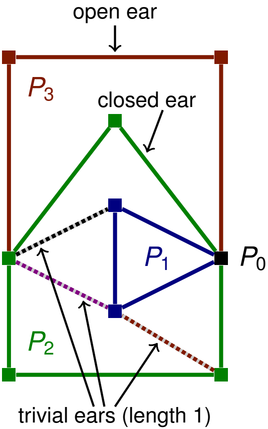

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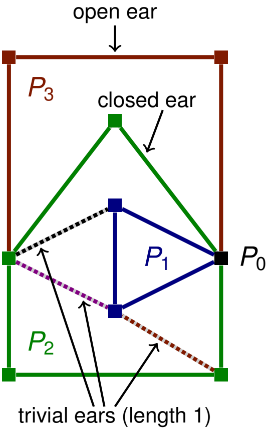

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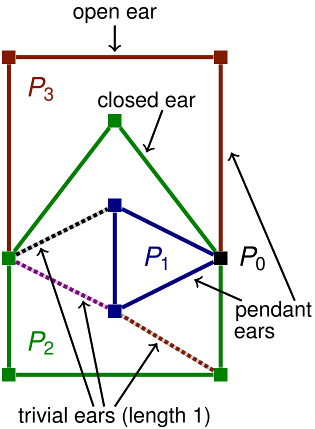

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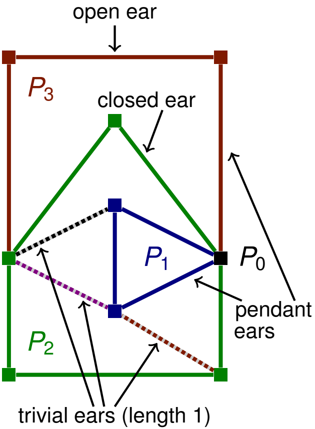

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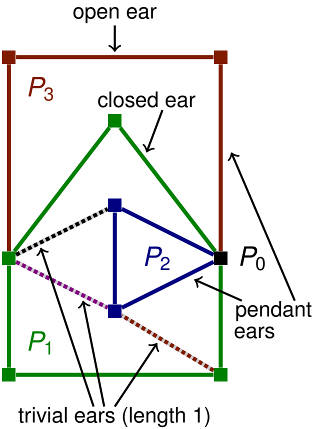

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Ear-decompositions with fewest even ears

For a 2-edge-connected graph G, let $\varphi(G)$ denote the minimum number of even ears in an ear-decomposition of G.

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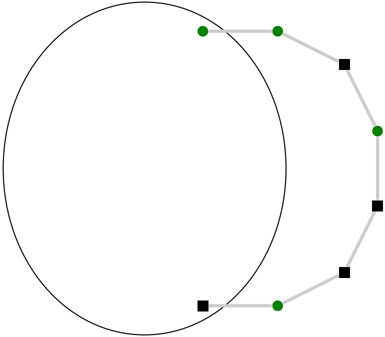

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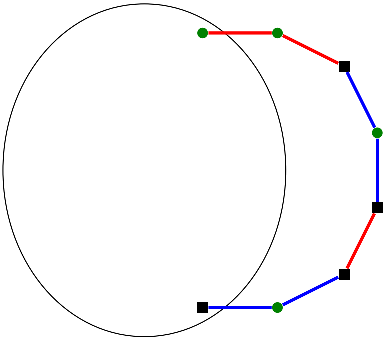

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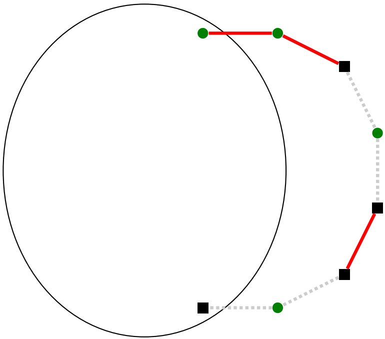

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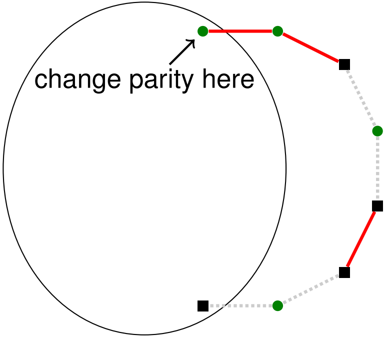

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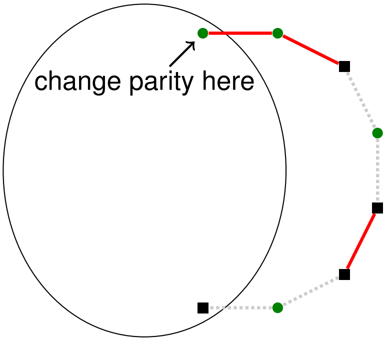

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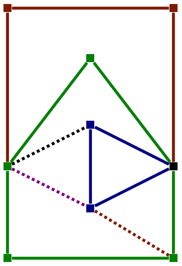
- This yields a T-join with $\leq \frac{1}{2}(|V(G)| - 1 + k_{\text{even}})$ edges

□

Ear-decompositions for 2ECSS

Simple algorithm for 2ECSS: delete all trivial ears.



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where k_{even} is the number of even ears,
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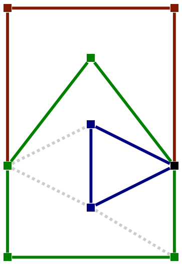

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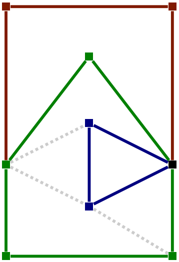

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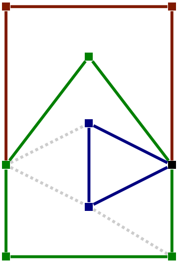

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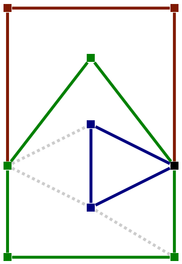

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This immediately yields a $\frac{3}{2}$-approximation for 2ECSS

(was improved to $\frac{17}{12}$ by Cheriyan, Sebő and Szigeti [2001])

New algorithm for 2ECSS

- Compute a nice ear-decomposition.
- Optimize short ears so that they serve best for connectivity.

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Alternative yields an 2ECSS with at most $\frac{5}{4}L + \frac{1}{2}\pi$ edges.

$\longrightarrow$ The better of the two 2ECSSs has at most $\frac{4}{3}L$ edges.

Nice ear-decompositions

An ear-decomposition is called nice if

(i) the number of even ears is minimum,

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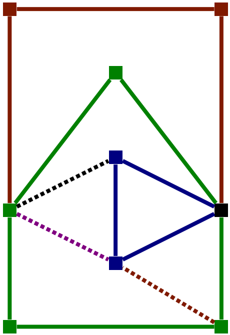

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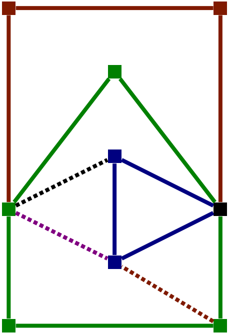

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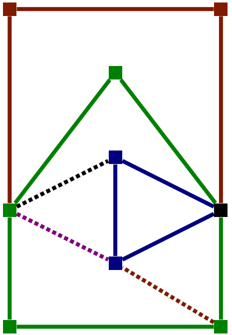

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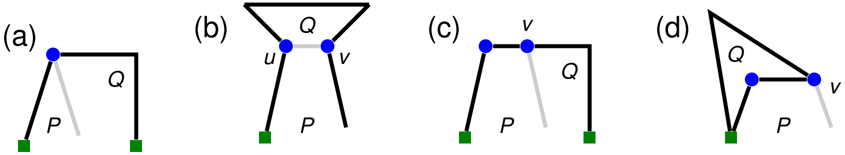
Sketch of Proof:

- Compute an ear-decomp. with fewest even ears (Frank [1993])
- Subdivide one edge of each even ear (⇒ factor-critical graph)
- Compute an open odd ear-decomp. (Lovász, Plummer [1986])
- Undo subdivisions ⇒ open ear-decomp. with fewest even ears
- Replace non-pendant short ears
- Replace adjacent short ears

□

Sketch of proof (some details)

- Replace non-pendant short ears



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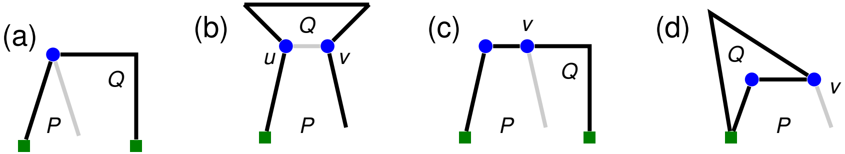

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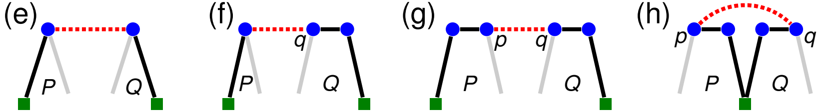
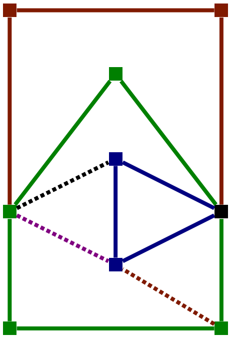


Optimizing short ears

- Adding all short ears leaves some number of connected components



Recall: An ear-decomposition is called nice if

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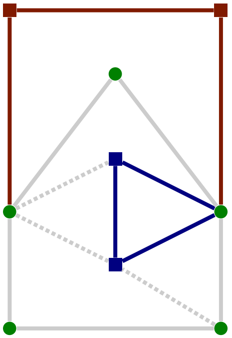

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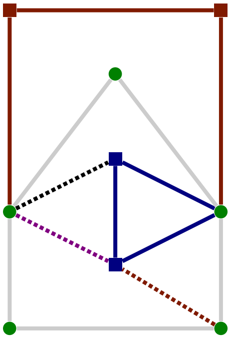

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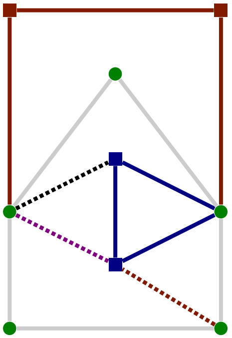

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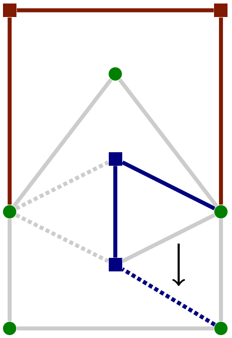

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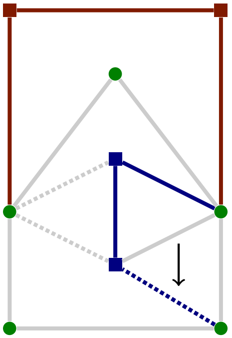

Note: Replacing some short ears by other ears (with the same internal vertices) will maintain a nice ear-decomposition.

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Eardrums and earmuffs

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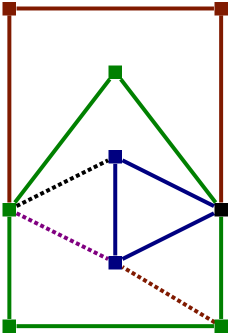

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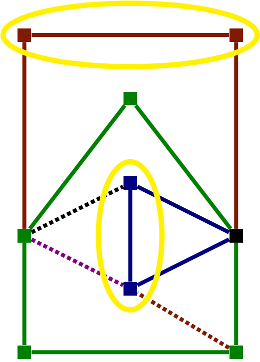

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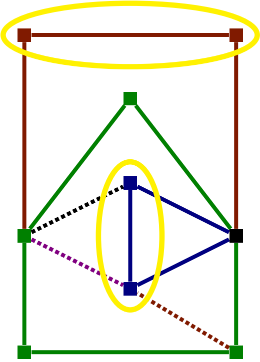

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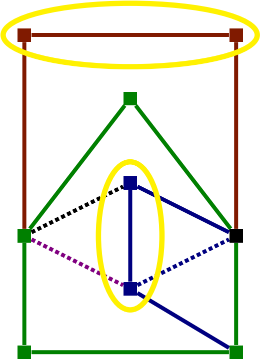
Let M be an eardrum. For $f \in M$ let $\mathcal{P}_f$ be the set of paths with internal vertices f.

Eardrums and earmuffs



An eardrum is the set of components of an induced subgraph with maximum degree 1.

Example: the sets of internal vertices of short ears in a nice ear-decomposition.

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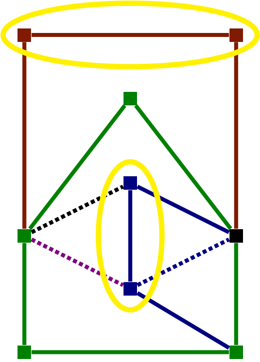
An earmuff is a set of paths $\{P_f : f \in F\}$ with $F \subseteq M$, $P_f \in \mathcal{P}_f$ for $f \in F$, and $(V(G), \bigcup_{f \in F} E(P_f))$ is a forest.

Theorem

A maximum earmuff can be computed in polynomial time.

Eardrums and earmuffs



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A maximum earmuff can be computed in polynomial time.

Call the maximum $\mu(G, M)$.

First solution: matroid intersection

- Represent each path $P \in \mathcal{P}_f$ ($f \in M$) by the set e_P of its two endpoints; let $E_f := \{e_P : P \in \mathcal{P}_f\}$.
- Let r be the rank function of the cycle matroid of the complete graph on $V(G)$.

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Theorem (Rado [1942])

Let E be a finite set and r the rank function of a matroid on E. Let $E_1, E_2, \ldots, E_k \subseteq E$. Then

$$\max\{r(\{e_1, \ldots, e_k\}) : e_i \in E_i \ (i = 1, \ldots, k)\} = \min\left\{r\left(\bigcup_{i \in I} E_i\right) + k - |I| : I \subseteq \{1, \ldots, k\}\right\}.$$

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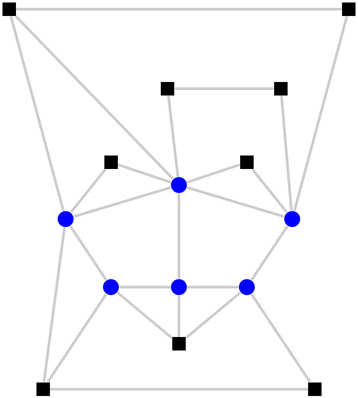
Note: Special case of matroid intersection.

Earmuff maximization: example

- ■ vertex in V_M
- ● vertex in $V(G) \setminus V_M$



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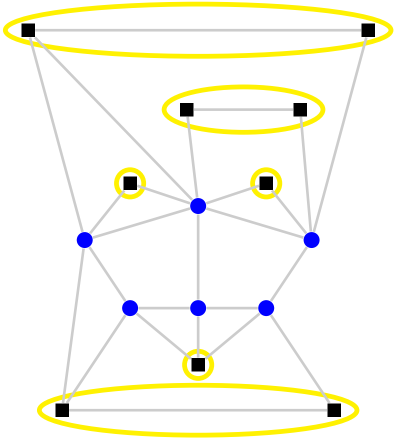

○ eardrum

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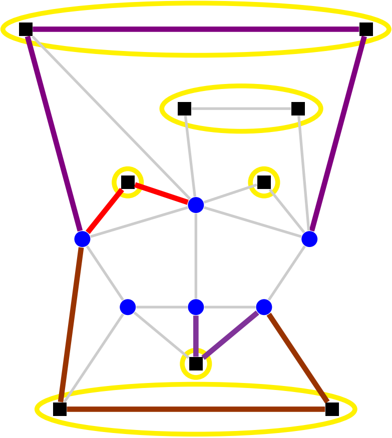
maximum earmuff

other edges

eardrum



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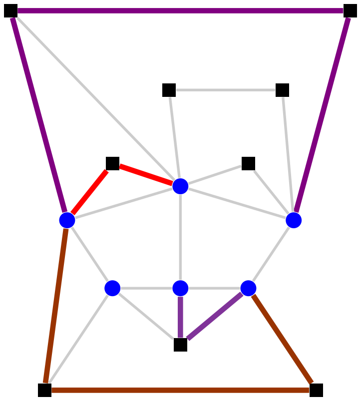

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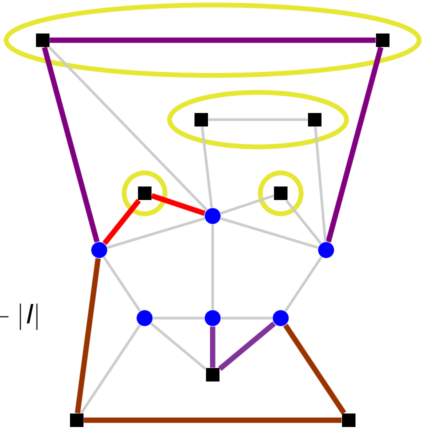

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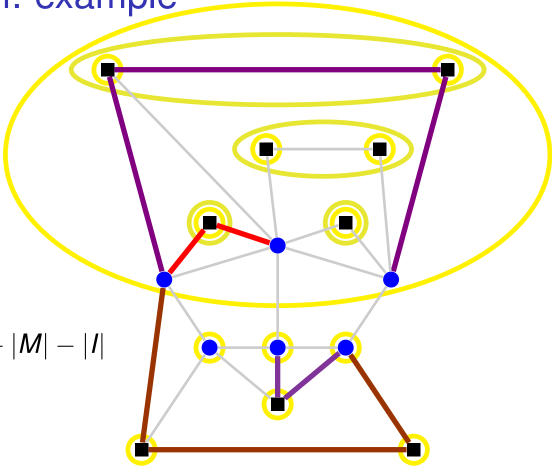
cuts in dual solution



No edge belongs to more than 2 cuts. Number of cuts $\geq$
$|V(G)| + |I| - r(\bigcup_{i \in I} E_i) - 1 \; = \; |V(G)| - 1 + |M| - \mu(G, M)$

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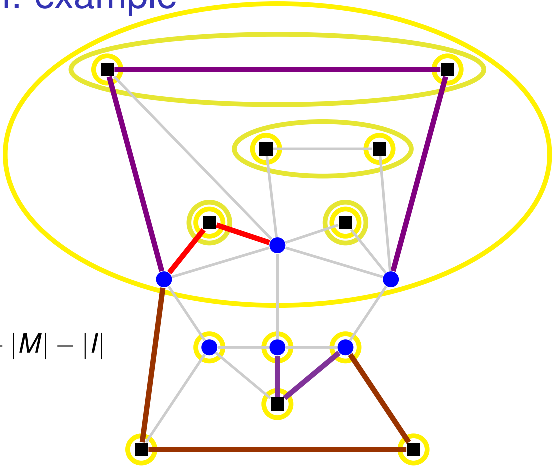

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Theorem

Any 2ECSS has at least $L_\mu := |V(G)| - 1 + |M| - \mu(G,M)$ *edges.* $\square$

New algorithm for 2ECSS

- Compute a nice ear-decomposition.
- Optimize short ears so that they serve best for connectivity.

Note: number of even ears is minimum, all short ears are pendant

- Take all edges of pendant ears.
- Add edges to obtain connectivity.
- Add edges to correct parity.

Alternatively:

- Take all edges of nontrivial ears.

Theorem

The new algorithm yields a tour with at most $\frac{3}{2}L - \pi$ edges, where L is a lower bound on the number of edges in any 2ECSS, and π is the number of pendant ears (after optimization).

Alternative yields an 2ECSS with at most $\frac{5}{4}L + \frac{1}{2}\pi$ edges.

$\longrightarrow$ The better of the two 2ECSSs has at most $\frac{4}{3}L$ edges.

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Proof: Since $|V_0| \leq |V(G)| + \varphi_\pi(G) - 2\pi_{\text{short}} - 4\pi_{\text{long}}$, correcting parity needs at most $\frac{1}{2}L_\varphi - \pi - \pi_{\text{long}}$ edges. $\square$

New algorithm for TSP

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New algorithm for TSP

- Compute a nice ear-decomposition.
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Theorem

Mömke-Svensson yields a tour with at most $\frac{4}{3}L + \frac{2}{3}\pi$ edges.

$\longrightarrow$ The better of the two tours has at most $\frac{7}{5}L$ edges.

The Mömke-Svensson lemma

Definition (Mömke and Svensson [2011])

Let G be a 2-vertex-connected graph.
A removable pairing in G consists of a set R of removable edges and a set of pairwise disjoint pairs of elements of R such that

- deleting any edge set $S \subseteq R$ that contains at most edge out of each pair does not disconnect the graph
- the two edges of any pair share a vertex, and this vertex is incident to another edge

Theorem (Mömke and Svensson [2011])

Given a 2-vertex-connected graph G and a removable pairing $(R, \mathcal{P})$. Then one can find a tour with at most $\frac{4}{3}|E(G)| - \frac{2}{3}|R|$ edges in polynomial time.

Mömke-Svensson applied to ear-decompositions

Theorem

Given a 2-vertex-connected graph G with an ear-decomposition in which all ears are nontrivial. Then one can find a tour with at most $\frac{4}{3}(|V(G)| - 1) + \frac{2}{3}\pi$ *edges in polynomial time.*

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Proof: We define the removable pairing as follows:

- For each pendant ear, R will contain exactly one of its edges.
- For each non-pendant ear, R will contain a pair of edges incident to an internal vertex that is endpoint of another ear.

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So $|R| = 2(|E(G)| - (|V(G)| - 1)) - \pi$.

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The Mömke-Svensson lemma yields a tour with at most $\frac{4}{3}|E(G)| - \frac{2}{3}|R|$ edges. $\square$

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Theorem (Mömke and Svensson [2011])

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Given a 2-vertex-connected graph G and a removable pairing $(R, \mathcal{P})$. Then one can find a tour with at most $\frac{4}{3}|E(G)| - \frac{2}{3}|R|$ edges in polynomial time.

Proof: Find an odd join J containing at most one edge of each pair.
Add a second copy of each edge in $J \setminus R$.
Delete the edges in $J \cap R$.
We get a tour with $|E(G)| + c(J)$ edges,
where $c(e) = -1$ for $e \in R$ and $c(e) = 1$ for $e \notin R$.

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Construct G' as follows. For each pair P with edges $\{u, v\}, \{v, w\}$, add a vertex v_P and an edge $\{v_P, v\}$ of weight 0, and replace the two edges in P by $\{u, v_P\}, \{v_P, w\}$.

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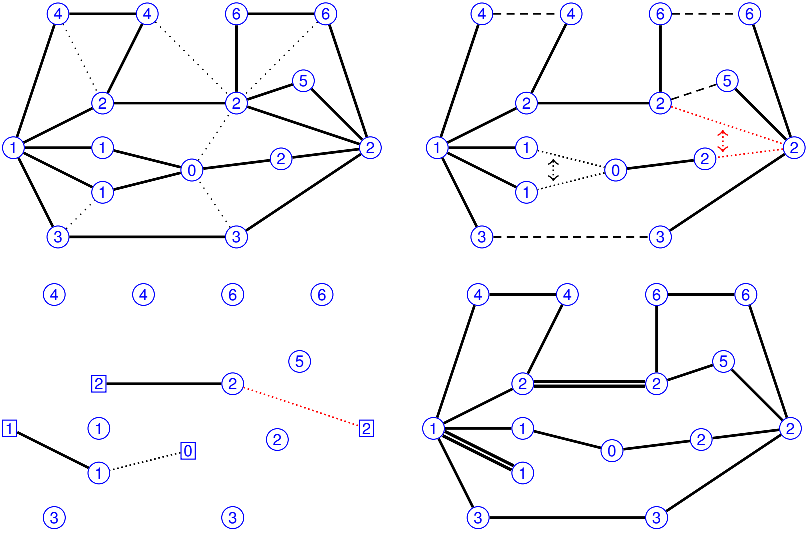
The all-$\frac{1}{3}$-vector is in the odd join polytope of G', and in its face defined by $x(\delta(v_P)) = 1$ for all pairs P.
Hence there is an odd join J as required with
$c(J) \leq \frac{1}{3}c(E(G')) = \frac{1}{3}|E(G)| - \frac{2}{3}|R|$. $\square$

Example: application of Mömke-Svensson lemma



New algorithm for TSP

- Compute a nice ear-decomposition.
- Optimize short ears so that they serve best for connectivity.
- Delete all 1-ears. In each of the resulting blocks:
- Take all edges of pendant ears.
- Add edges to obtain connectivity.
- Add edges to correct parity.

Alternatively:

- Apply lemma of Mömke-Svensson.

Theorem

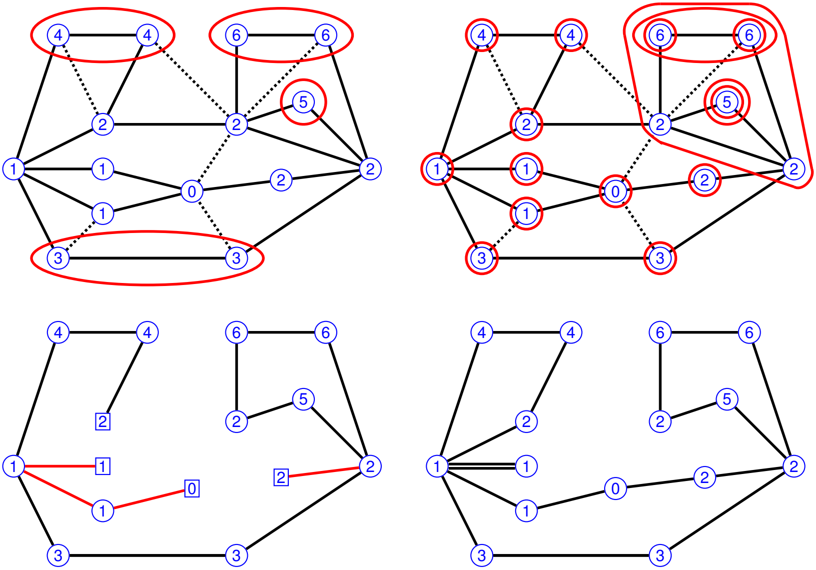
In each block, this algorithm yields a tour with at most $\frac{3}{2}L - \pi$ edges, where L is a lower bound on the number of edges in any 2ECSS, and π is the number of pendant ears (after optimization).

Theorem

Mömke-Svensson yields a tour with at most $\frac{4}{3}L + \frac{2}{3}\pi$ edges.

$\longrightarrow$ The better of the two tours has at most $\frac{7}{5}L$ edges.

Example: Shorter tour by nicer ears



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- Add edges to correct parity.

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Mömke-Svensson yields a tour with at most $\frac{4}{3}L + \frac{2}{3}\pi$ edges.

$\longrightarrow$ The better of the two tours has at most $\frac{7}{5}L$ edges.

New algorithm for connected-T-joins

- Compute a nice ear-decomposition.
- Optimize clean ears so that they serve best for connectivity.

(Clean ears are short ears without an internal vertex in T.)

- Take all edges of clean ears.
- Apply ear induction to pendant but not clean ears.
- Add edges to obtain connectivity.
- Add edges to correct parity.

Alternatively:

- Apply ear induction to all ears.

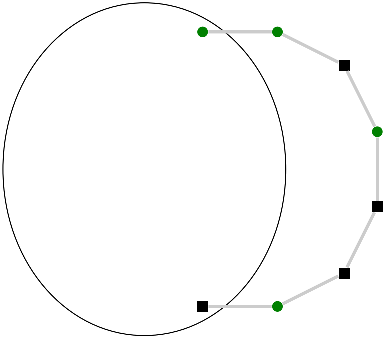
Theorem

The new algorithm yields a connected-T-join with at most $\frac{3}{2}L + \frac{1}{2}\varphi(G) - \pi$ edges, where L is a lower bound and π is the number of pendant ears (after optimization).

Alternative yields at most $\frac{3}{2}L - \frac{1}{2}\varphi(G) + \pi$ edges.

$\longrightarrow$ The better of the two has at most $\frac{3}{2}L$ edges.

Tool for connected-T-joins: ear induction



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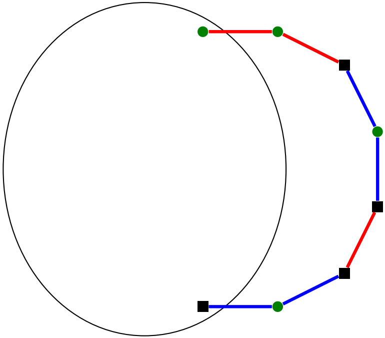
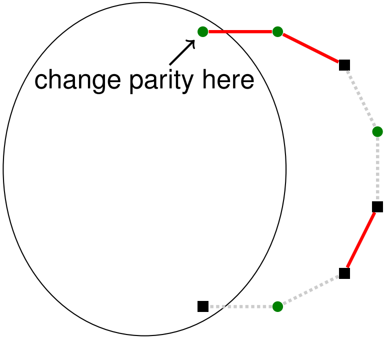


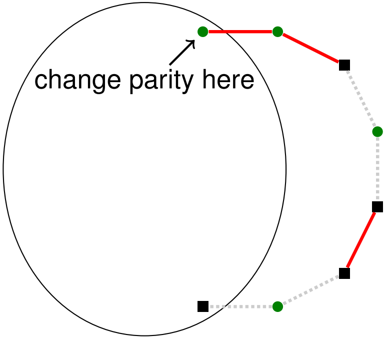
- Split pendant ear at vertices in T (that have wrong parity so far)

Tool for connected-T-joins: ear induction



- Split pendant ear at vertices in T (that have wrong parity so far)
- Take smaller part for obtaining a T-join

Tool for connected-T-joins: ear induction



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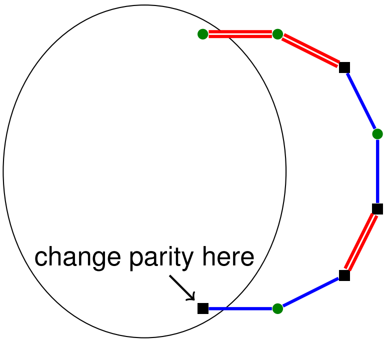
This yields a

- T-join with $\leq \frac{1}{2}(|V(G)| - 1 + k_{\text{even}})$ edges

Tool for connected-T-joins: ear induction



- Split pendant ear at vertices in T (that have wrong parity so far)
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- Double smaller part for obtaining a connected-T-join

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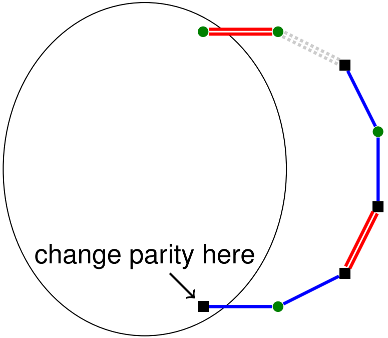

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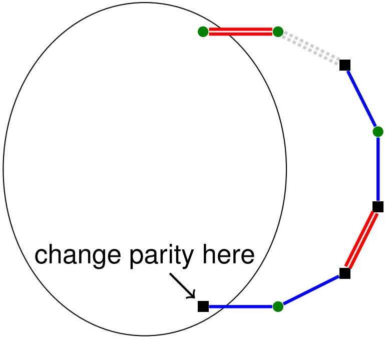

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This yields a

- T-join with $\leq \frac{1}{2}(|V(G)| - 1 + k_{\text{even}})$ edges
- connected T-join with $\leq \frac{3}{2}(|V(G)| - 1) + \pi_{\text{clean}} - \frac{1}{2}k_{\text{even}} - k_{\text{odd}}$ edges

$\leq \frac{3}{2}(|V(G)|-1) + \frac{1}{2}k_{\text{even}} - \pi_{\text{notclean}}$ and $\leq \frac{3}{2}(|V(G)|-1) - \frac{1}{2}k_{\text{even}} + \pi$

New algorithm for connected-T-joins

- Compute a nice ear-decomposition.
- Optimize clean ears so that they serve best for connectivity.

Clean ears are short ears without an internal vertex in T.

- Take all edges of clean ears.
- Apply ear induction to pendant but not clean ears.
- Add edges to obtain connectivity.
- Add edges to correct parity.

Alternatively:

- Apply ear induction to all ears.

Theorem

The new algorithm yields a connected-T-join with at most $\frac{3}{2}L + \frac{1}{2}\varphi(G) - \pi$ edges, where L is a lower bound and π is the number of pendant ears (after optimization).

Alternative yields at most $\frac{3}{2}L - \frac{1}{2}\varphi(G) + \pi$ edges.

$\longrightarrow$ The better of the two has at most $\frac{3}{2}L$ edges.

Using nicer ears, more generally and formally

Definition

Let G be a graph with a nice ear-decomposition, $T \subseteq V(G)$, $|T|$ even.
An ear is clean if it is short and T contains none of its internal vertices.
Let M contain for each clean ear the set of its internal vertices.

Theorem

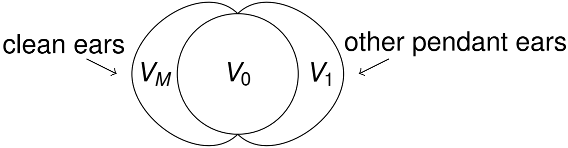
Let G, T, M be as above. Suppose the given ear-decomposition contains a maximum earmuff for M (of cardinality μ).
Then a connected-T-join of cardinality at most $L_\mu + \frac{1}{2}L_\varphi - \pi$ can be constructed in $O(|V(G)|^3)$ time.

Recall:

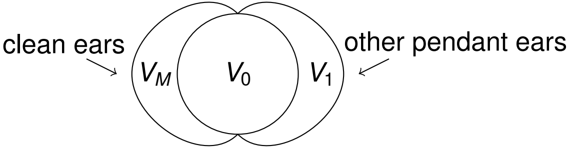
$$\begin{aligned} \pi &= \text{number of pendant ears} \\ L_\varphi &= |V(G)| - 1 + \varphi(G) \\ L_\mu &= |V(G)| - 1 + |M| - \mu \end{aligned}$$

L_μ is a lower bound on the optimum (in fact, on the LP value).
L_φ is a lower bound if $T = \emptyset$.

Proof

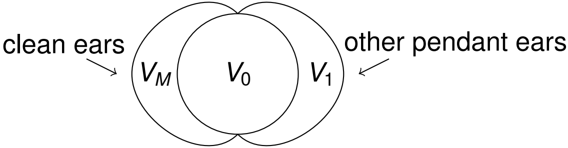


Proof



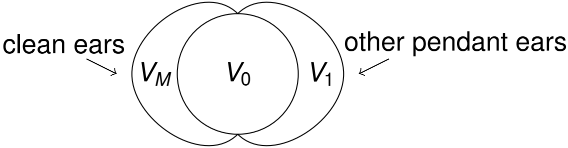
- take all edges of clean ears $\frac{3}{2}|V_M| + \frac{1}{2}\varphi_M$

Proof



- take all edges of clean ears $\qquad \frac{3}{2}|V_M| + \frac{1}{2}\varphi_M$
- add edges of $G[V_0]$ so that
  $(V_M \cup V_0, E_1 \cup E_2)$ is connected $\qquad |V_0| - 1 - \mu$

Proof



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- apply ear induction to non-clean pendant ears $\qquad \frac{3}{2}|V_1| + \frac{1}{2}\varphi_1 - (\pi - |M|)$

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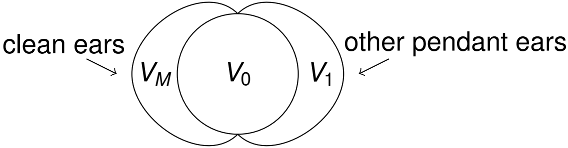

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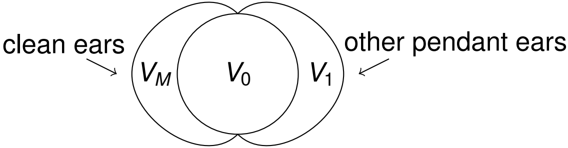
- Correct parities on $G[V_0]$ $\quad \frac{1}{2}(|V_0| + \varphi_0 - 1)$

Proof



► take all edges of clean ears	$\frac{3}{2}\lvert V_M\rvert + \frac{1}{2}\varphi_M$
► add edges of $G[V_0]$ so that $(V_M \cup V_0, E_1 \cup E_2)$ is connected	$\lvert V_0\rvert - 1 - \mu$
► apply ear induction to non-clean pendant ears	$\frac{3}{2}\lvert V_1\rvert + \frac{1}{2}\varphi_1 - (\pi - \lvert M\rvert)$
► Correct parities on $G[V_0]$	$\frac{1}{2}(\lvert V_0\rvert + \varphi_0 - 1)$

Adding up, using $\varphi(G) = \varphi_0 + \varphi_1 + \varphi_M$, yields

$$\frac{3}{2}(|V(G)| - 1) + |M| - \mu + \frac{1}{2}\varphi(G) - \pi \quad = \quad L_\mu + \frac{1}{2}L_\varphi - \pi$$

□

Second solution to earmuff maximization

Let M be a eardrum with $V_M \cap T = \emptyset$ (e.g., contain for each clean ear the set of its internal vertices).

- Let $U = V(G) \setminus V_M$, where $V_M = \bigcup M$.
- Represent $\mathcal{P}_f$ ($f \in M$) by the set U_f of its endpoints.
- Sufficient to find a maximum cardinality subset $F \subseteq M$ with a forest representative system $(e_f)_{f \in F}$ of $(U_f)_{f \in F}$, i.e.,
  - $e_f \in \binom{U_f}{2}$ for all $f \in F$,
  - $e_f \neq e_{f'}$ for $f \neq f'$, and
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Theorem (≈ Lovász [1970])

This maximum is $\mu =$

$$\min\left\{ |M| - \sum_{W \in \mathcal{W}} \big(|\{f \in M : U_f \subseteq W\}| - (|W| - 1) \big) : \mathcal{W} \textit{ is a partition of } U \right\}.$$

(This holds for any finite sets U, M, $(U_f)_{f \in M}$ with $\emptyset \neq U_f \subseteq U$.)

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Note: We have an algorithm with running time $O(|V(G)||E(G)|)$

Lower bounds and LP relaxations

Theorem (Cheriyan, Sebő, Szigeti [2001])

$$L_\varphi := |V(G)| - 1 + \varphi(G) \le \mathrm{LP}(G) :=$$

$$\min\left\{x(E(G)) : x \in \mathbb{R}^{E(G)}_{\geq 0},\ x(\delta(W)) \geq 2 \textit{ for all } \emptyset \neq W \subset V(G)\right\}$$

Proof (Sketch): Frank's theorem $\Rightarrow$ T $\Rightarrow$ 2-packing of T-cuts. $\square$

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Theorem

$$L_\mu \;:=\; |V(G)| - 1 + |M| - \mu \;\leq\; \mathrm{LP}(G, T) \;:=$$

$$\min\Big\{x(E(G)) \;:\; x \in \mathbb{R}_{\geq 0}^{E(G)},$$
$$x(\delta(W)) \geq 2 \textit{ for all } \emptyset \neq W \subset V(G) \textit{ with } |W \cap T| \textit{ even},$$
$$x(\delta(\mathcal{W})) \geq |\mathcal{W}| - 1 \textit{ for all partitions } \mathcal{W} \textit{ of } V(G)\Big\}$$

Proof (Sketch): 2-packing of cuts, including the partition from min-max theorem for forest representation systems. $\square$

Summary of Results

We obtained an improved approximation ratio of:

- $\frac{7}{5}$ for Graphic TSP
- $\frac{3}{2}$ for Connected-T-join
- $\frac{4}{3}$ for 2ECSS

- All algorithms combinatorial, running time $O(|V(G)|^3)$
- These bounds are tight.
- These are also upper bounds on the integrality ratios of the natural LPs for unit weights.

Open problems

- improve approximation ratios (to $\frac{4}{3}$ for Graphic TSP?)
- extend to weights (general metrics)
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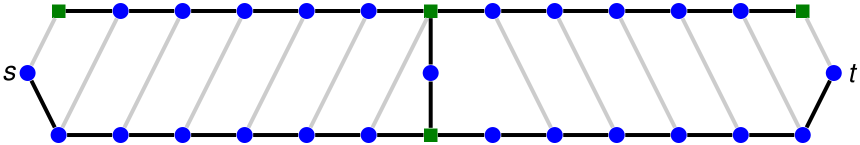
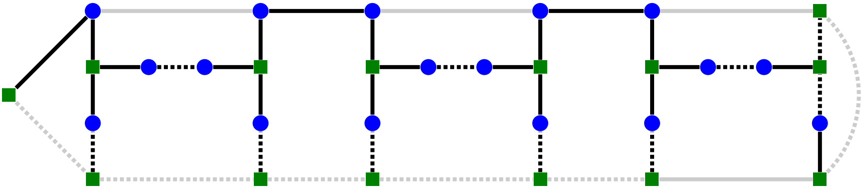
Tight example for connected-T-join



$|V(G)| = 8k + 5$ (Here $k = 3$.)
$T = \{s, t\}$
$OPT = 8k + 4$
$\varphi(G) = 2$
$\pi = 1 = \frac{1}{2}\varphi(G)$.

Algorithm computes solution with $12k + 6$ edges.

Tight example for graphic TSP



$|V(G)| = OPT = 10k + 1$ (Here $k = 3$.)

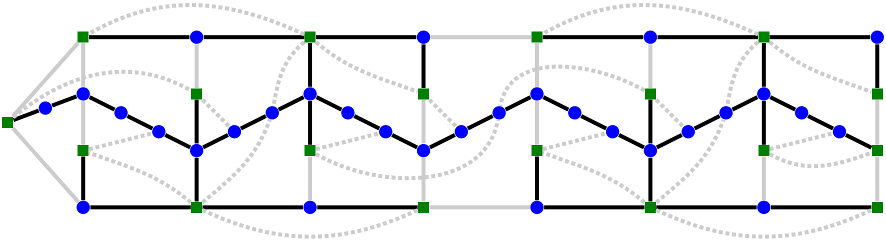
$\varphi(G) = 0$

$L = 10k$

$\pi = k = \frac{1}{10}L$.

Algorithm computes solution with $14k$ edges.

Tight example for 2ECSS



$L = |V(G)| = OPT = 24k$ (Here $k = 2$.)
$\varphi(G) = 1$
$\pi = 4k = \frac{1}{6}L.$

Algorithm computes solution with $32k - 1$ edges.