Research Institute for Discrete Mathematics
Chip Design
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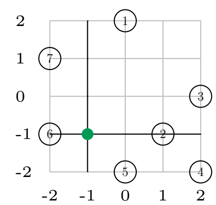
Prof. Dr. S. Hougardy
D. Rotter, M. Sc.

Solution to Exercise 4.3 (d)

Let T_I denote the initial set T and $n := |T_I|$. For $z \in \mathbb{R}_{>0}$ let $T_z := \{t \in T : ||t-r||_1 \geq z\}$. We construct the tree in a sweep-line manner starting with $z = \max\{||t-r||_1 : t \in T_I\} + 1$ and ending with $z = 0$. At each step of the algorithm we try to include Steiner points which have distance z from r.
During the whole algorithm we store T_z in an array sorted circularly as follows:

1. elements p s.t. $p - r$ lies in the 1st quadrant ordered by x-coord. in non-decr. order
2. elements p s.t. $p - r$ lies in the 2nd quadrant ordered by x-coord. in non-incr. order
3. elements p s.t. $p - r$ lies in the 3rd quadrant ordered by x-coord. in non-incr. order
4. elements p s.t. $p - r$ lies in the 4th quadrant ordered by x-coord. in non-decr. order

We say that two points $t_1, t_2 \in T_z$ are *neighbors*, if they are adjacent in T_z w.r.t. that ordering (or if one of them is the first and the other is the last element).



We store the set $S_z := (T \backslash T_z) \cup \{\text{med}(t_1, t_2, r) : t_1, t_2 \in T_z \text{ neighbors}\}$ in a 2-heap with key $||. - r||_1$ such that we can determine the element with maximum key very fast. By convention, we say that in case of ties, DELETEMAX returns elements of $T \backslash T_z$ first.

Claim: If $t_1, t_2 \in T$ s.t. $||\text{med}(t_1, t_2, r) - r||_1 = z$, then there are *neighbors* $t_1', t_2' \in T_z$ such that $||\text{med}(t_1', t_2', r) - r||_1 \geq z$.
Proof: Assume that the claim is wrong. We select a counter-example t_1, t_2 such that t_1 and t_2 are non-neighbors which are as close as possible w.r.t. the ordering of T_z. W.l.o.g., we may assume that $r = (0,0)$ and t_1 is in the 1st quadrant. If t_2 is in the 3rd quadrant, $\text{med}(t_1, t_2, r) = r$ and the claim is trivial. Let t_2 be in the 2nd quadrant. We have: $\text{med}(t_1, t_2, r) = (\min\{x(t_1), x(t_2)\}, 0) = (z, 0)$. Let $t \in T_z$ come after t_1 but before t_2 in the ordering of T_z. Since $x(t_1) \leq x(t)$ if t is in the first quadrant and $x(t_2) \leq x(t)$ if t is in the second quadrant, it holds that $\min\{x(t_1), x(t_2)\} \leq \min\{x(t_1), x(t)\}$ and hence, $||\text{med}(t_1, t, r) - r||_1 \geq ||\text{med}(t_1, t_2, r) - r||_1 = z$.
Let t_2 be in the 4th quadrant. We have: $\text{med}(t_1, t_2, r) = (0, \min\{y(t_1), y(t_2)\}) = (0, z)$. Since t_1 and t_2 are non-neighbors, t_1 is not the first or t_4 is not the last element of T_z. W.l.o.g. let the first case be true (the other case is similar). Let t be the first element of T_z. If $y(t) \geq z$, we are done. Otherwise, $0 \leq y(t) < z \leq y(t_1)$ and thus, $\text{med}(t, t_1, r) = t$ which concludes the case. Finally, assume that both t_1 and t_2 are in the first quadrant, w.l.o.g. $x(t_1) \leq x(t_2)$. Let t be any vertex between t_1 and t_2. If $y(t_1) \leq y(t)$ or $y(t) \leq y(t_2)$, we are done. Otherwise, $y(t_1) > y(t) > y(t_2)$ and $||\text{med}(t_1, t, r) - r||_1 = x(t_1) + y(t) \geq x(t_1) + y(t_2) = ||\text{med}(t_1, t_2, r) - r||_1 = z$. □

Assume, we have already computed T_z and S_z for some $z > 0$ and that we have already included all Steiner points with distance larger than z from r. Let $t_1, t_2 \in T$ such that $||\text{med}(t_1, t_2, r) - r||_1$ is maximum. There are two cases:

Case (i): $||med(t_1, t_2, r) - r||_1 < z$. Then, we cannot include a Steiner point with distance z from r. All further Steiner points have distance at least $z' := \max\{||s - r||_1 : s \in S_z\}$ which is exactly the maximum key of the heap. We can decrease z to z', include all sinks with distance z' to the root to T_z and update S_z appropriately. Let T' be the set of new elements in T_z. The number of elements we have to include to S_z (i.e. points which arise by joining an element of T' with a neighbor) as well as the points which we have to remove from S_z (i.e. T' and points which arise by joining elements in T_z which are no longer neighbors) is at most $c \cdot |T'|$ for a constant c.
We can determine T' by $|T'|$ DELETEMAX operations. We can determine the correct position of a new element in an ordered list in $\mathcal{O}(\log(|T_z|)) = \mathcal{O}(\log(n))$ time. Deleting from or inserting into a 2-heap takes $\mathcal{O}(\log(n))$ time. Hence, this iteration can be performed in $\mathcal{O}(|T'| \cdot \log(n))$ time.

Case (ii): $||med(t_1, t_2, r) - r||_1 = z$. By the claim, t_1 and t_2 can be chosen to be neighbors in T_z and hence, $\text{med}(t_1, t_2, r)$ is an element of S_z with maximum key. The key of each sink in S_z is strictly smaller than z.
Hence, we can determine $t' := \text{med}(t_1, t_2, r)$ (or a median-point with the same distance to r) by a DELETEMAX operation. We include t' to T_z but delete t_1 and t_2. We delete all (at most 3) points arising by joining t_1 or t_2 with a neighbor from S_z (t' is among them) and include all (at most 2) points arising by joining t' with a neighbor. All of these operations need $\mathcal{O}(\log(n))$ time.

When the heap is empty, $|T_z| = 1$ and we can connect the remaining sink with r.

Since all of the sets T' as defined in case (i) are pairwise disjoint subsets of T_I, all case (i)-operations in total take $\mathcal{O}(n \cdot \log(n))$ time.
All applications of case (ii) result in the insertion of a Steiner point. Let S be the set of Steiner points. By construction, $|\delta(s)| = 3$ for each $s \in S$ and $|\delta(t)| = 1$ for $t \in T_I \cup \{r\}$. Thus,

$$(|S| + n + 1) - 1 = \tfrac{1}{2} \sum_{v \in S \cup T_I \cup \{r\}} |\delta(v)| = \tfrac{1}{2} \cdot (3|S| + n + 1)$$

which implies $|S| = n - 1$. Hence, all case (ii)-operations in total require $\mathcal{O}(n \cdot \log(n))$ time.