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Exercise Set 3

Exercise 3.1. Let $m \in \mathbb{N}$. Show that a circuit C for $f_{0,m}$ over the basis $\{\wedge, \vee\}$ with depth $D(C) \leq \log_2 m + \log_2 \log_2 m + \mathcal{O}(1)$ and size $S(C) \in \mathcal{O}(m \log m)$ can be computed in time $\mathcal{O}(m^3)$.

(5 points)

Exercise 3.2. Consider the following recursively defined family of Boolean functions $f_{2n,m} \in B_{2n+m+1}$ $(n, m \in \mathbb{N})$:

$$\begin{aligned} f_{0,0}(x_0) &= x_0, \\ f_{0,m}(x_0, \dots, x_m) &= x_0 \wedge f^*_{0,m-1}(x_1, \dots, x_m) && (m \geq 1), \\ f_{2n,m}(x_0, \dots, x_{2n+m}) &= x_0 \wedge f_{2n-2,m}(x_2, \dots, x_{2n+m}) && (m \geq 0, n \geq 1), \end{aligned}$$

where $f^*(x_1, \dots, x_n) := \neg f(\bar{x}_1, \dots, \bar{x}_n) \in B_n$ is the dual function of a function $f \in B_n$. Prove the following split equation

$$\begin{aligned} f_{2n,2k+m+1}(x_0, x_1, \dots, x_{2n+2k+m+1}) &= f_{2n,2k}(x_0, \dots, x_{2n+2k}) \\ &\quad \wedge f^*_{2k,m}(x_{2n+1}, x_{2n+2}, \dots, x_{2n+2k+m+1}). \end{aligned}$$

(5 points)

Exercise 3.3. Let $n = 2^k$ for $k \in \mathbb{N}$ and a, b two n-bit numbers representing $|a|, |b| \in \mathbb{N}$. Define $f^n \in B_{2n,2n}$ as $f^n(a, b) := |a| \cdot |b|$ i.e. the product of two naturals.

(a) A *bit-shift* is a multiplication by 2^i for $i \in \mathbb{N}$. Show that $|a| \cdot |b|$ can be expressed in terms of at most 3 non-bit-shift multiplications of $\frac{n}{2}$-bit numbers, 6 additions of $2n$-bit numbers, and several bit-shifts.

(b) Show $S(n) \leq 3 \cdot S(\frac{n}{2}) + cn$ and $D(n) \leq D(\frac{n}{2}) + d \cdot \log_2 n$ for constants c and d.

(c) Let $\Omega := \{\wedge, \vee, \oplus\}$. Show $S_\Omega(f^n) = \mathcal{O}(n^{\log_2 3})$ and $D_\Omega(f^n) = \mathcal{O}(\log_2^2 n)$ for circuits with fanout 2.

(1 + 1 + 3 points)

Exercise 3.4. For a Boolean circuit C with inputs $1, \dots, n$ and arrival times $t_i \in \mathbb{N}$ $(i = 1, \dots, n)$, its delay is defined as its depth after prepending a path with t_i circuits to input i $(i = 1, \dots, n)$.

(a) Show that for n inputs with arrival times $t_i \in \mathbb{N}$ $(i = 1, \dots, n)$ there are n-ary AND, OR or XOR circuits over B_2 with delay $d \in \mathbb{N}$ if and only if

$$\sum_{i=1}^{n} 2^{t_i - d} \leq 1.$$

(b) Provide an algorithm that finds such a circuit in $\mathcal{O}(n \log n)$ time.

(3 + 2 points)

Deadline: April 28, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss22/chipss22_ex.html`

In case of any questions feel free to contact me at blankenburg@or.uni-bonn.de.