Research Institute for Discrete Mathematics
Universität Bonn

Linear and Integer Optimization
Assignment Sheet 9
Inofficial English Translation

1. Consider the following optimization problem:

$$\begin{array}{ll} \min & \frac{c^t x+d}{f^t x+g} \\ \text{s.t.} & Ax \leq b \\ & \|x\|_\infty \leq R \end{array}$$

where $c, f \in \mathbb{Q}^n$, $d, g, R \in \mathbb{Q}$, $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$. You may assume that $f^t x+g > 0$ and $c^t x+d > 0$ for any $x \in \mathbb{R}^n$ with $\|x\|_\infty \leq R$ and that there is a feasible solution. Show that for any $\epsilon > 0$ there is a polynomial-time algorithm computing a feasible solution x^* with $\frac{c^t x^*+d}{f^t x^*+g} \leq \mathrm{OPT}(1+\epsilon)$ where OPT is the value of an optimum solution. (5 points)

2. Let $K \subseteq \mathbb{R}^n$ be an r-R-sandwiched convex set, $c \in \mathbb{R}^n$, $\delta = \sup\{c^t x \mid x \in K\}$, and $0 < \epsilon < \delta$. Moreover, let $U = \{x \in K \mid c^t x \geq \delta - \epsilon\}$. Prove that

$$\text{volume}\,(U) \geq \left(\frac{\epsilon}{2\|c\|R|}\right)^{n-1} r^{n-1}\frac{1}{n^n}\frac{\epsilon}{2\|c\|}\frac{1}{n}.$$

(5 points)

3. Determine numbers k and l as small as possible such that a given feasible and bounded linear program $\max\{c^t x \mid Ax \leq b\}$ with $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$ and $c \in \mathbb{Q}^n$ can be solved in time $O((m+n)^k(\text{size}(A)+\text{size}(b)+\text{size}(c))^l)$ by applying the Ellipsoid Algorithm. (6 points)

4. Consider the following primal-dual pair of linear programs: (P): $\max\{c^t x \mid Ax + s = b, s \geq 0\}$ and (D): $\min\{b^t y \mid A^t y = c, y \geq 0\}$ with $A \in \mathbb{R}^{m\times n}$. Assume that both LPs are feasible. By strict complementary slackness, there is a partitioning $\{1,\ldots,m\} = B \dot\cup N$ such that for $i \in B$ there is an optimum dual solution y^* with $y_i^* > 0$ and for $i \in N$ there is an optimum primal solution x^*, s^* with $s_i^* > 0$. Describe a linear program such that any optimum solution of it directly gives you the set B and N. (4 points)

Due date: Tuesday, June 14, 2022, before the lecture in the lecture hall.