Near-Optimum Global Routing with Coupling, Delay Bounds, and Power Consumption

Jens Vygen

Research Institute for Discrete Mathematics, University of Bonn
Lennéstraße 2, 53113 Bonn, Germany
vygen@or.uni-bonn.de

Abstract. We propose the first theoretical approach to global routing that takes coupling between adjacent wires, bounds on delays along critical paths, and overall capacitance (power consumption) into account. It consists of an efficient combinatorial fully polynomial approximation scheme to a fractional relaxation, followed by randomized rounding. The overall deviation from the optimum can be bounded. The model could also be used for routing traffic flows with congestion-dependent travel times.

1 Introduction

Because of the enormous size of VLSI routing instances, detailed routing is usually preceded by global routing, defining an area for each net to which the path search in detailed routing will be restricted.

But global routing is also needed for another reason: not every placement allows a feasible routing. Moreover, a feasible routing may be possible only with intolerable detours. It is important to detect infeasible routing instances, and – if possible – a certificate of infeasibility, faster than by actually trying to find a feasible routing.

In its simplest version, the global routing problem amounts to packing Steiner trees, connecting given terminal sets, in an undirected graph with edge capacities. For this problem, Albrecht [2001a,2001b] described an approach that is very successful in practice. It consists of solving a fractional relaxation by a combinatorial fully polynomial approximation scheme (extending earlier work on multicommodity flows) and then applying randomized rounding.

However, neither this nor any other approach (see Hu and Sapatnekar [2001] for a survey) optimizes the main design objectives: timing, power consumption, and manufacturing yield. Minimizing the total length of the Steiner trees is not important, but minimizing a weighted sum of capacitances is: this is equivalent to minimizing power consumption. Delays on critical paths also depend on the capacitances of their nets. Even when assuming constant wire width, however, the capacitance can no longer be assumed to be proportional to the length, because coupling between neighbouring wires plays an increasingly important role. Often small detours are better than the densest possible packing. Spreading the wires also helps to improve the yield. Recent papers on global routing (Xu et al. [2003], Jing et al. [2003], Ho et al. [2003]) propose heuristics partially addressing these objectives, but none of them can obtain any performance guarantee.

We describe the first global routing algorithm, with a provable performance guarantee, that takes coupling, timing, and power consumption into account directly. The yield is also improved as a by-product. Our algorithm extends earlier work on multicommodity flows, fractional global routing, and randomized rounding. After describing the problem in section 2, our main result will be a fully polynomial approximation scheme for a fractional relaxation, presented in section 3. In section 4 we show how to make randomized rounding work. Finally, in section 5, we conclude with remarks on related problems in routing traffic flow and general min-max resource sharing problems.

2 Global Routing in VLSI Layout

For this paper it does not matter whether or not detailed routing is done grid-based, and whether or not we allow diagonal wires. In any case, the chip area with its (currently approximately 8) routing planes is partitioned into regions (in the simplest case by an axis-parallel grid). The regions will form the vertices of the global routing graph. Adjacent regions are joined by an edge, whose capacity indicates how many wires of unit width can join the two regions while observing all routing constraints. A new promising approach to computing accurate capacities fast is described by Müller [2002].

For each net, we consider the set of regions that contain at least one of its pins. If a pin has shapes intersecting more than one region, we can assign it to one of them heuristically (e.g. the one that is closest to the center of gravity of the net, or based on the solution to the corresponding Group Steiner Tree Problem). If a net has all pins in one region, it is ignored in global routing. Usually it is wise to wire nets with very short connections beforehand and have them as blockages in the subsequent routing process. This allows more accurate capacity estimations. Of course, one may reroute such nets later if deemed necessary.

In the following, let G be the global routing graph, with edge capacities $u : E(G) \to \mathbb{R}$ and lengths $l : E(G) \to \mathbb{R}_+$. Let $\mathcal{N}$ be the set of nets. For each $N \in \mathcal{N}$ we have a set $\mathcal{Y}_N$ of feasible Steiner trees. For example, we may restrict attention to Elmore-delay-optimal Steiner trees for certain critical nets, or impose length bounds. For the following it is not even important to know what nets and Steiner trees are. The only assumption we make is that each Steiner tree Y is associated with an edge set $E(Y)$, and we can find for any $N \in \mathcal{N}$ and any $\psi : E(G) \to \mathbb{R}_+$ an element $Y \in \mathcal{Y}_N$ with $\sum_{e\in E(Y)} \psi(e)$ (almost) minimum sufficiently fast. In fact, in most cases $\mathcal{Y}_N$ will simply contain all Steiner trees for N in G.

Moreover, we assume to have constants $w(N, e) \in \mathbb{R}_+$ describing the width of a wire for net N that uses edge e. For an integer programming formulation, we choose a Steiner tree for each net by binary variables $x_{N,Y}$ ($N \in \mathcal{N}$, $Y \in \mathcal{Y}_N$).

Then a traditional version of the global routing problem can be formulated as follows:

$$\begin{array}{lll}
\min & \sum_{N\in\mathcal{N}} c(N) \sum_{e\in E(G)} l(e) \sum_{Y\in\mathcal{Y}_N : e\in E(Y)} x_{N,Y} & \\
s.t. & \sum_{N\in\mathcal{N}} \sum_{Y\in\mathcal{Y}_N : e\in E(Y)} w(N,e) x_{N,Y} \leq u(e) & \text{for } e \in E(G) \\
& \sum_{Y\in\mathcal{Y}_N} x_{N,Y} = 1 & \text{for } N \in \mathcal{N} \\
& x_{N,Y} \in \{0,1\} & \text{for } N \in \mathcal{N} \text{ and } Y \in \mathcal{Y}_N
\end{array} \tag{1}$$

To decide whether an instance of this problem has a feasible solution is NP-complete even in the case of a planar grid with unit capacities and two-terminal nets only (Raghavan [1986]). This remains true if we impose length bounds, even if all nets must be routed shortest possible (Vygen [1994]).

Before extending this classical global routing model, let us make some comments on previous work. For a comprehensive survey on various global routing approaches we refer to Hu and Sapatnekar [2001].

The only approach for which a good theoretical performance guarantee can be proved (under reasonable assumptions) goes back to Raghavan and Thompson [1987,1991], who proposed to solve an LP relaxation first, and then use randomized rounding to obtain an integral solution whose maximum capacity violation can be bounded. This also proves a bound on the integrality gap, as we shall see in Section 4. A similar approach with a more complicated rounding step has been proposed by Karp et al. [1987].

Together with polynomial-time linear programming algorithms, randomized rounding yields an approximation algorithm with respect to the objective of minimizing the maximum relative congestion

$$\max_{e\in E(G)} \frac{1}{u(e)} \sum_{N\in\mathcal{N} : e\in E(Y_N)} w(N,e). \tag{2}$$

However, for instances corresponding to current complex chips (with millions of edges and nets), all exact algorithms are too slow for solving even the LP relaxation optimally. Therefore fast approximation algorithms are interesting. Even better, there exist combinatorial fully polynomial approximation schemes, i.e. algorithms that compute a feasible (fractional) solution whose LP value is within a factor of $1+\epsilon$ of the optimum, and whose running time is bounded by a polynomial in $|V(G)|$ and $\frac{1}{\epsilon}$ (and is less than needed for general linear programming). Most research has focused on the special case where $w \equiv 1$, every net has exactly two elements, and $\mathcal{Y}_N$ contains all paths connecting N. This is the Edge-Disjoint Paths Problem, and its fractional relaxation is known as the Multicommodity Flow Problem. The first combinatorial fully polynomial approximation scheme for multicommodity flows was developped by Shahrokhi and Matula [1990]; see Karakostas [2002] for the latest improvement and more references.

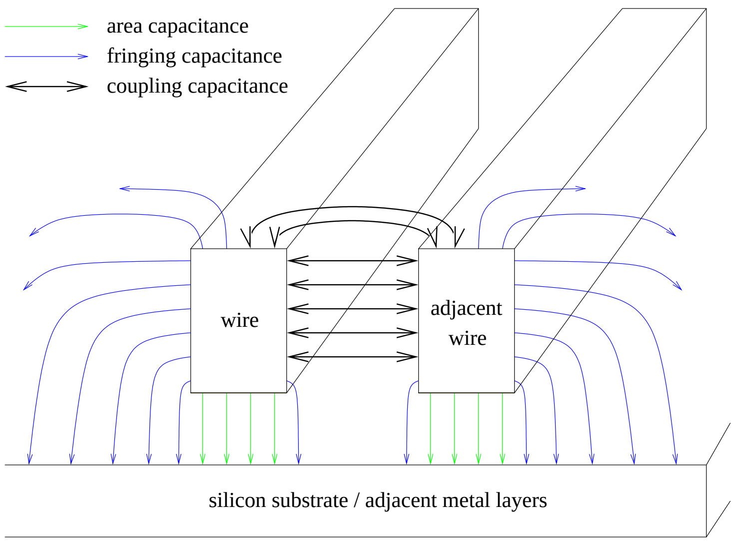

Fig. 1.

Approximation algorithms for multicommodity flow problems have been applied to global routing first by Carden, Li and Cheng [1996]. Later, Albrecht [2001a,2001b] applied one of the latest approximation algorithms, due to Garg and Könemann [1998], to the LP relaxation of (1), and combined it with ideas of several others in order to obtain a better running time in practice (without increasing the theoretical worst-case running time). We will now generalize this approach to take the important design objectives into account.

The power consumption of a chip – as far as influenced by routing – is proportional to the weighted sum of all capacitances, weighted by switching activities. The capacitance of a net consists of area capacitance (proportional to length times width), fringing capacitance (proportional to length) and coupling capacitance (proportional to the length if adjacent wires exist); see Figure 1. Of course, the coupling capacitance also depends on the distance of the two parallel wires. In older technologies, the impact of coupling was quite small and often ignored. This is no longer feasible with currently used deep-submicron technologies.

We model the capacitance as follows. Let $l(e, N)$ be the maximum capacitance of a wire along global routing edge e, assuming parallel wires running at both sides at minimum distance. As we fix the width of each pair of net and edge in advance, these numbers are known. Now we assume that extra space assigned to a wire reduces the capacitance linearly, up to some maximum space. If we reserve space $w(N, e) + y(e, N)s(e, N)$, where $0 \leq y(e, N) \leq 1$, we have a capacitance $l(e, N) - y(e, N)v(e, N)$. This is a simplification in two respects: first, coupling capacitance does not depend linearly on the distance, and secondly, it may not always be possible to have all wires with $y(e, N) = 1$ next to each other.

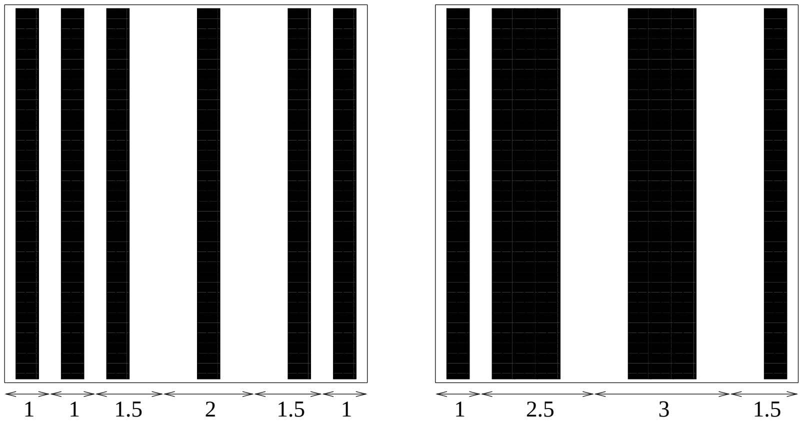

Fig. 2.

Often we will have $v(e,N) = v(e,N')$ and $s(e,N) = s(e,N')$ for any two nets N and N'. However, the more general model causes no additional difficulties. Figure 2 shows a channel corresponding to a global routing edge e of capacity $u(e) = 8$ with two sets of wires using it. In the first case (to the left) we have six wires of unit width, in the second case (to the right) we have two unit width wires and two double width wires. In each case, the wires use up 6 units of capacity. For maximum spreading (one extra unit free space between each pair of adjacent wires we would need capacity 12 in the first case and 10 in the second one. Assuming $s(e,N) = v(e,N) = 1$ for all nets N, the two layouts shown the nets (from left to right) get capacitance reductions $0, 0, \frac{1}{2}, 1, \frac{1}{2}, 0$ and $0, \frac{1}{2}, 1, \frac{1}{2}$. Note that in the presence of two wires with capacitance reduction 0 and 1 we must have at least two wires with coupling capacitance in between. In our model, this is ignored, and also blockages, pin shapes and vias are ignored. Nevertheless quite accurate results can be obtained.

Minimizing power consumption rather than total netlength will automatically lead to a better spreading of the wires, and thus to increased yield. In addition, we can also take timing constraints into account. On the one hand, the set $\mathcal{Y}_N$ of feasible Steiner trees may exclude those that lead to intolerable detours. On the other hand, we can impose upper bounds on weighted sums of capacitances for sets of nets. This can be applied, for example, to the set of nets in each critical path. One possible methodology is to do a first static timing analysis based on the assumption that every net has some expected capacitance (only Steiner trees below this bound will be in the set $\mathcal{Y}_N$), enumerate all paths that have negative slack under this assumption, compute the sensitivity of their nets to capacitance changes, and use these values to translate the delay bound to a bound on the weighted sum of capacitances for each path. To compute reasonable expected capacitances one could apply weighted slack balancing (see Albrecht et al. [2002] for an efficient algorithm) using delay sensitivity and congestion information. The details will have to be developped by future research. We are very confident, however, that the following model will prove useful in practice.

GLOBAL ROUTING PROBLEM

Instance:

- An undirected graph G with capacities $u : E(G) \to \mathbb{R}_+$.
- A set $\mathcal{N}$ of so-called nets, and a family $\mathcal{Y}_N$ of Steiner trees for each $N \in \mathcal{N}$. We assume that we can find a $Y \in \mathcal{Y}_N$ with minimum $\sum_{e \in E(Y)} \psi(e)$ fast for any $\psi : E(G) \to \mathbb{R}_+$.
- Wire widths $w : \mathcal{N} \times E(G) \to \mathbb{R}_+$ and possible extra space $s : E(G) \times \mathcal{N} \to \mathbb{R}_+$.
- Maximum capacitances $l : E(G) \times \mathcal{N} \to \mathbb{R}_+$, and coupling contributions $v : E(G) \times \mathcal{N} \to \mathbb{R}_+$.
- A family $\mathcal{M}$ of subsets of $\mathcal{N}$, where $\mathcal{N} \in \mathcal{M}$, bounds $U : \mathcal{M} \to \mathbb{R}_+$ and weights $c(M, N) \in \mathbb{R}_+$ for $N \in M \in \mathcal{M}$.

Task: Find a $Y_N \in \mathcal{Y}_N$ and numbers $0 \le y(e, N) \le 1$ for $e \in Y_N$, for each net $N \in \mathcal{N}$, such that

- $\displaystyle\sum_{N \in \mathcal{N} : e \in E(Y_N)} (w(N, e) + s(e, N) y(e, N)) \le u(e)$ for each edge e,
- $\displaystyle\sum_{N \in M} c(M, N) \sum_{e \in E(Y_N)} (l(e, N) - v(e, N) y(e, N)) \le U(M)$ for $M \in \mathcal{M}$,
- minimizing $\displaystyle\sum_{N \in \mathcal{N}} c(\mathcal{N}, N) \sum_{e \in E(Y_N)} (l(e, N) - v(e, N) y(e, N))$.

The assumption that we can find optimum Steiner trees fast is justified as almost all nets in a reasonable practical instance have a small number of pins (less than 10). Note that a dynamic programming algorithm exists that runs in $O(3^k n + 2^k (m + n \log n))$ time, where k is the number of terminals, $n = |V(G)|$ and $m = |E(G)|$ (Erickson, Monma and Veinott [1987]).

3 Fractional Global Routing

First observe that it suffices to compute a feasible solution as we can impose an arbitrary upper bound $U(\mathcal{N})$ on

$$\sum_{N \in \mathcal{N}} c(\mathcal{N}, N) \sum_{e \in E(Y_N)} (l(e, N) + v(e, N) y(e, N)),$$

and apply binary search to find the optimum value of $U(\mathcal{N})$. Then we look for a feasible solution (λ, x, y) to the following LP, where x is integral and $\lambda = 1$. As this is hard, we first solve the following LP approximately and then (in Section 4) apply randomized rounding.

$$
\begin{aligned}
&\min \lambda \\
&\text{s.t.} \sum_{Y \in \mathcal{Y}_N} x_{N,Y} = 1 && \text{for } N \in \mathcal{N} \\
&\quad \sum_{N \in M} c(M,N) \left(\sum_{Y \in \mathcal{Y}_N} \sum_{e \in E(Y)} l(e,N) x_{N,Y} - \sum_{e \in E(G)} v(e,N) y_{e,N} \right) \leq \lambda U(M) && \\
&&& \text{for } M \in \mathcal{M} \\
&\quad \sum_{N \in \mathcal{N}} \left(\sum_{Y \in \mathcal{Y}_N : e \in E(Y)} w(N,e) x_{N,Y} + s(e,N) y_{e,N} \right) \leq \lambda u(e) && \text{for } e \in E(G) \\
&\quad y_{e,N} \leq \sum_{Y \in \mathcal{Y}_N : e \in E(Y)} x_{N,Y} && \text{for } e \in E(G),\ N \in \mathcal{N} \\
&\quad y_{e,N} \geq 0 && \text{for } e \in E(G),\ N \in \mathcal{N} \\
&\quad x_{N,Y} \geq 0 && \text{for } N \in \mathcal{N},\ Y \in \mathcal{Y}_N
\end{aligned} \tag{3}
$$

The corresponding dual LP is:

$$
\begin{aligned}
&\max \sum_{N \in \mathcal{N}} z_N \\
&\text{s.t.} \sum_{e \in E(G)} u(e)\omega_e + \sum_{M \in \mathcal{M}} U(M)\mu_M = 1 \\
&\quad z_N \leq \sum_{e \in E(Y)} \left(l(e,N) \sum_{M \in \mathcal{M} : N \in M} c(M,N)\mu_M + w(N,e)\omega_e - \chi_{N,e} \right) && \\
&&& \text{for } N \in \mathcal{N},\ Y \in \mathcal{Y}_N \\
&\quad \chi_{e,N} \geq v(e,N) \sum_{M \in \mathcal{M} : N \in M} c(M,N)\mu_M - s(e,N)\omega_e && \text{for } e \in E(G),\ N \in \mathcal{N} \\
&\quad \chi_{e,N} \geq 0 && \text{for } e \in E(G),\ N \in \mathcal{N} \\
&\quad \omega_e \geq 0 && \text{for } e \in E(G) \\
&\quad \mu_M \geq 0 && \text{for } M \in \mathcal{M}
\end{aligned} \tag{4}
$$

The dual LP enables us to compute lower bounds on the optimum LP value:

Lemma 1. *Let $\omega_e \in \mathbb{R}_+$ $(e \in E(G))$ and $\mu_M \in \mathbb{R}_+$ $(M \in \mathcal{M})$. Let* $\psi_{N,e} :=$

$$
\min_{\delta \in \{0,1\}} \left((l(e,N) - \delta v(e,N)) \sum_{M \in \mathcal{M} : N \in M} c(M,N)\mu_M + (w(N,e) + \delta s(e,N))\omega_e \right). \tag{5}
$$

Then

$$
\frac{\sum_{N \in \mathcal{N}} \min_{Y \in \mathcal{Y}_N} \sum_{e \in E(Y)} \psi_{N,e}}{\sum_{e \in E(G)} u(e)\omega_e + \sum_{M \in \mathcal{M}} U(M)\mu_M} \tag{6}
$$

is a lower bound on the optimum LP value of (3) and (4).

Proof: Set $\chi_{e,N} := \max\left\{0, v(e,N)\sum_{M\in\mathcal{M}:N\in M} c(M,N)\mu_M - s(e,N)\omega_e\right\}$ and

$$z_N := \min_{Y\in\mathcal{Y}_N} \sum_{e\in E(Y)} \left(l(e,N) \sum_{M\in\mathcal{M}:N\in M} c(M,N)\mu_M + w(N,e)\omega_e - \chi_{N,e} \right). \quad (7)$$

After dividing all variables by $\sum_{e\in E(G)} u(e)\omega_e + \sum_{M\in\mathcal{M}} U(M)\mu_M$, we get a feasible solution (z,μ,ω,χ) of (4). $\square$

Now we can present our approximation algorithm. It uses four parameters $0 < \epsilon, \epsilon_1, \epsilon_2 \le 1$, and $t \in \mathbb{N}$, which will be determined later to prove bounds on the approximation guarantee and running time. Moreover, we assume to have a lower bound $\mathcal{L}_N$ on the length $\sum_{e\in E(Y)}(l(e,N) - v(e,N))$ of any Steiner tree $Y \in \mathcal{Y}_N$ $(N \in \mathcal{N})$. Often a nontrivial lower bound can be computed in constant time using distances of the terminals.

FRACTIONAL GLOBAL ROUTING ALGORITHM

Input: An instance of the GLOBAL ROUTING PROBLEM with $\mathcal{N} = \{1,\dots,k\}$.

Output: A feasible solution to (3).

① Set $\omega_e := \frac{1}{u(e)}$ for $e \in E(G)$ and $\mu_M := \frac{1}{U(M)}$ for $M \in \mathcal{M}$.
Set $x_{i,Y} := 0$ for $i = 1,\dots,k$, $Y \in \mathcal{Y}_i$
Set $y_{e,i} := 0$ for $e \in E(G)$ and $i = 1,\dots,k$.
Set $Y_i := \emptyset$ for $i = 1,\dots,k$.

② **For** $p := 1$ to t **do**:
  For $i := 1$ **to** k **do**:
    Let $\psi_{i,e}$ be defined as in (5).
    If $Y_i = \emptyset$ or $\sum_{e\in E(Y_i)} \psi_{i,e} > (1+\epsilon_1)z_i$ **then**:
      Let $Y_i \in \mathcal{Y}_i$ with $\sum_{e\in E(Y_i)} \psi_{i,e} \le (1+\epsilon_2) \min_{Y\in\mathcal{Y}_i} \sum_{e\in E(Y)} \psi_{i,e}$.
      Set $z_i := \sum_{e\in E(Y_i)} \psi_{i,e}$.
    Set $x_{i,Y_i} := x_{i,Y_i} + 1$.
    For $e \in E(Y_i)$ **do**:
      If $v(e,i)\sum_{M\in\mathcal{M}:i\in M} c(M,i)\mu_M < s(e,N)\omega_e$ **then** $\delta_e := 0$
        else $\delta_e := 1$.
      Set $y_{e,i} := y_{e,i} + \delta_e$.
      Set $\omega_e := \omega_e e^{\epsilon \frac{w(i,e)+\delta_e s(e,i)}{u(e)}}$.
      For $M \in \mathcal{M}$ with $i \in M$ **do**:
        Set $\mu_M := \mu_M e^{\epsilon c(M,i) \frac{l(e,i)-\delta_e v(e,i)}{U(M)}}$.
        For $j \in M$ **do**:
          Set $z_j := z_j + (1+\epsilon_2)\mathcal{L}_j c(M,j)\mu_M \left(e^{\epsilon c(M,i)\frac{l(e,i)-\delta_e v(e,i)}{U(M)}} - 1\right)$.

③ Set $x_{i,Y} := \frac{1}{t} x_{i,Y}$ for $i = 1, \ldots, k$ and $Y \in \mathcal{Y}_i$.
Set $y_{e,i} := \frac{1}{t} y_{e,i}$ for $e \in E(G)$ and $i = 1, \ldots, k$.

We can show that this algorithm computes an approximate solution to (3). The following proof (and the above algorithm) uses and extends ideas of Young [1995], Garg and Könemann [1998], Fleischer [2000], and Albrecht [2001a,2001b].

Theorem 2. *Let an instance of the* GLOBAL ROUTING PROBLEM *be given, and let* $m = |E(G)|$. *Let* $\Lambda \geq 1$, $\Lambda \geq \frac{c(M,i) \sum_{e \in E(Y)} l(e,i)}{U(M)}$ *for* $i \in M \in \mathcal{M}$ *and* $Y \in \mathcal{Y}_i$, *and* $\Lambda \geq \frac{w_{i,e} + s(e,i)}{u(e)}$ *for* $i \in \mathcal{N}$ *and* $e \in E(G)$.

Let $\epsilon_1, \epsilon_2 \geq 0$ *and* $0 < \epsilon < 1$. *Let* $\epsilon' := \epsilon(1 + \frac{3}{4}\Lambda\epsilon)(1 + \epsilon_1)(1 + \epsilon_2)$. *Assume that* $\epsilon'\lambda^* < 1$, *where* λ^* *is the LP value of (3). Let* $t \in \mathbb{N}$.

Let $\rho := \frac{\epsilon'}{\epsilon(1-\epsilon'\lambda^*)\left(1 - \frac{\log(m+|\mathcal{M}|)}{\epsilon t \lambda^*}\right)}$.

Then the above algorithm with parameters $\epsilon, \epsilon_1, \epsilon_2, t$ *computes a* ρ*-approximate solution to (3). More precisely, the vectors* x *and* y *computed by the algorithm and* $\rho\lambda^*$ *constitute a feasible solution to (3).*

Proof: Let $\mu_M^{(p,j)}, \omega_e^{(p,j)}, Y_j^{(p,j)}, \delta_e^{(p,j)}$ be the values of the variables $\mu_M, \omega_e, Y_j, \delta_e$ at the end of the j-th inner iteration within the p-th outer iteration of the algorithm ($e \in E(G), M \in \mathcal{M}$), and let $\psi_{i,e}^{(p,j)}$ be defined as in (5) with respect to $\mu_M^{(p,j)}$ and $\omega_e^{(p,j)}$ ($e \in E(G)$, $i, j = 1, \ldots, k$, $M \in \mathcal{M}$). (So in iteration (p, j) we use $\psi_{j,e}^{(p,j-1)}$ as edge lengths.) Let $\mu_M^{(p)} := \mu_M^{(p,k)}$ and $\omega_e^{(p)} := \omega_e^{(p,k)}$ be the values at the end of iteration p. We write $\mu_M^{(p,0)} := \mu_M^{(p-1)}$ etc., and $\mu_M^{(0)}$ is the value of μ_M after ①.

By construction we have $0 \leq y_{e,i} \leq \sum_{Y \in \mathcal{Y}_i : e \in E(Y)} x_{i,Y}$ for $e \in E(G)$, $i \in \{1, \ldots, k\}$, and $x_{i,Y} \geq 0$ for $i \in \{1, \ldots, k\}$, $Y \in \mathcal{Y}_i$. We will check for which value of λ the other constraints of (3) hold.

Initially (after ①) we have

$$\sum_{e \in E(G)} u(e)\omega_e^{(0)} + \sum_{M \in \mathcal{M}} U(M)\mu_M^{(0)} = m + |\mathcal{M}|. \tag{8}$$

We first estimate the increase of the y-variables. We have

$$\begin{aligned}
&\sum_{M \in \mathcal{M}} U(M)\mu_M^{(p,i)} \\
&= \sum_{M \in \mathcal{M}: i \notin M} U(M)\mu_M^{(p,i-1)} + \sum_{M \in \mathcal{M}: i \in M} U(M)\mu_M^{(p,i-1)} e^{\epsilon c(M,i) \sum_{e \in E(Y_i)} \frac{l(e,i) - \delta_e^{(p,i)} v(e,i)}{U(M)}} \\
&\leq \sum_{M \in \mathcal{M}} U(M)\mu_M^{(p,i-1)} \\
&\qquad + \epsilon(1 + \tfrac{3}{4}\Lambda\epsilon) \sum_{M \in \mathcal{M}: i \in M} \mu_M^{(p,i-1)} c(M,i) \sum_{e \in E(Y_i)} (l(e,i) - \delta_e^{(p,i)} v(e,i)),
\end{aligned} \tag{9}$$

because $\epsilon \leq 1$, $c(M,i) \sum_{e \in E(Y_i)} (l(e,i) - \delta_e^{(p,i)} v(e,i)) \leq \Lambda U(M)$ for $i \in M \in \mathcal{M}$, and $e^x \leq 1 + x + \frac{3}{4}x^2$ for $0 \leq x \leq 1$.

Similarly,

$$\begin{aligned}\sum_{e\in E(G)} & u(e)\omega_e^{(p,i)} \\ &= \sum_{e\notin E(Y_i)} u(e)\omega_e^{(p,i-1)} + \sum_{e\in E(Y_i)} u(e)\omega_e^{(p,i-1)} e^{\epsilon\frac{w(i,e)+\delta_e^{(p,i)} s(e,i)}{u(e)}} \\ &\le \sum_{e\in E(G)} u(e)\omega_e^{(p,i-1)} + \epsilon(1+\tfrac{3}{4}\Lambda\epsilon) \sum_{e\in E(Y_i)} (w(i,e)+\delta_e^{(p,i)} s(e,i))\omega_e^{(p,i-1)}.\end{aligned} \tag{10}$$

Combining (9) and (10) we get

$$\begin{aligned}\sum_{e\in E(G)} & u(e)\omega_e^{(p)} + \sum_{M\in\mathcal{M}} U(M)\mu_M^{(p)} \\ &\le \sum_{e\in E(G)} u(e)\omega_e^{(p-1)} + \sum_{M\in\mathcal{M}} U(M)\mu_M^{(p-1)} + \epsilon(1+\tfrac{3}{4}\Lambda\epsilon)\sum_{i=1}^{k}\sum_{e\in E(Y_i)} \psi_{i,e}^{(p,i-1)}.\end{aligned} \tag{11}$$

At any stage we have $z_i \le (1+\epsilon_2)\min_{Y\in\mathcal{Y}_i}\sum_{e\in E(Y)}\psi_{i,e}$: the ψ-variables never decrease, and if the z-variables increase (say by $(1+\epsilon_2)\alpha$), then $\sum_{e\in E(Y)}\psi_{i,e}$ increases by at least α for any $Y\in\mathcal{Y}_i$. We conclude:

$$\sum_{e\in E(Y_i)} \psi_{i,e}^{(p,i-1)} \le (1+\epsilon_1)z_i \le (1+\epsilon_1)(1+\epsilon_2)\min_{Y\in\mathcal{Y}_i}\sum_{e\in E(Y)}\psi_{i,e}^{(p)}. \tag{12}$$

Hence, combining (11) and (12),

$$\begin{aligned}\sum_{e\in E(G)} & u(e)\omega_e^{(p)} + \sum_{M\in\mathcal{M}} U(M)\mu_M^{(p)} \\ &\le \sum_{e\in E(G)} u(e)\omega_e^{(p-1)} + \sum_{M\in\mathcal{M}} U(M)\mu_M^{(p-1)} + \epsilon'\sum_{i=1}^{k}\min_{Y\in\mathcal{Y}_i}\sum_{e\in E(Y)} \psi_{i,e}^{(p)},\end{aligned} \tag{13}$$

where $\epsilon' := \epsilon(1+\frac{3}{4}\Lambda\epsilon)(1+\epsilon_1)(1+\epsilon_2)$.

By Lemma 1 and weak linear programming duality,

$$\lambda_{lb} := \frac{\sum_{i=1}^{k}\min_{Y\in\mathcal{Y}_i}\sum_{e\in E(Y)}\psi_{i,e}^{(p)}}{\sum_{e\in E(G)} u(e)\omega_e^{(p)} + \sum_{M\in\mathcal{M}} U(M)\mu_M^{(p)}} \tag{14}$$

is a lower bound on the optimum LP value λ^*. Assuming $\epsilon'\lambda^* < 1$, (13) yields

$$\begin{aligned}\sum_{e\in E(G)} & u(e)\omega_e^{(p)} + \sum_{M\in\mathcal{M}} U(M)\mu_M^{(p)} \\ &\le \frac{1}{1-\epsilon'\lambda_{lb}^{(p)}}\left(\sum_{e\in E(G)} u(e)\omega_e^{(p-1)} + \sum_{M\in\mathcal{M}} U(M)\mu_M^{(p-1)}\right).\end{aligned} \tag{15}$$

Combining (8) and (15) and setting $\lambda_{lb} := \max_{p=1}^{t} \lambda_{lb}^{(p)}$, we get

$$\begin{aligned}
\sum_{e \in E(G)} u(e)\omega_e^{(p)} + \sum_{M \in \mathcal{M}} U(M)\mu_M^{(p)} &\leq \frac{m + |\mathcal{M}|}{(1 - \epsilon'\lambda_{lb})^p} \\
&= (m + |\mathcal{M}|)\left(1 + \frac{\epsilon'\lambda_{lb}}{1 - \epsilon'\lambda_{lb}}\right)^p \\
&\leq (m + |\mathcal{M}|)e^{\frac{\epsilon'\lambda_{lb} p}{1 - \epsilon'\lambda_{lb}}}, \\
&\leq (m + |\mathcal{M}|)e^{\frac{\epsilon'\lambda_{lb} p}{1 - \epsilon'\lambda^*}},
\end{aligned} \tag{16}$$

as $1 + x \leq e^x$ for $x \geq 0$. For $p = t$ we get

$$\lambda_{lb} \geq \frac{1 - \epsilon'\lambda^*}{\epsilon' t} \ln\left(\frac{\mathcal{U}}{m + |\mathcal{M}|}\right), \tag{17}$$

where $\mathcal{U} := \sum_{e \in E(G)} u(e)\omega_e^{(t)} + \sum_{M \in \mathcal{M}} U(M)\mu_M^{(t)}$.

For an upper bound on the lengths, let $M \in \mathcal{M}$. At termination we have

$$\begin{aligned}
&\sum_{i \in M} c(M, i)\left(\sum_{Y \in \mathcal{Y}_i} \sum_{e \in E(Y)} l(e, i)x_{i,Y} - \sum_{e \in E(G)} v(e, i)y_{e,i}\right) \\
&\quad = \frac{1}{t}\sum_{p=1}^{t} \sum_{i \in M} \sum_{e \in Y_i^{(p)}} c(M, i)(l(e, i) - \delta_e^{(p,i)} v(e, i)) \\
&\quad = \frac{U(M)}{\epsilon t} \ln\left(\prod_{p=1}^{t} \prod_{i \in M} \prod_{e \in Y_i^{(p)}} e^{\epsilon c(M,i)\frac{l(e,i) - \delta_e^{(p,i)} v(e,i)}{U(M)}}\right) \\
&\quad = \frac{U(M)}{\epsilon t} \ln\left(U(M)\mu_M^{(t)}\right) \\
&\quad \leq \frac{U(M)}{\epsilon t} \ln \mathcal{U}
\end{aligned} \tag{18}$$

Finally, we have for $e \in E(G)$:

$$\begin{aligned}
&\sum_{i=1}^{k} \left(\sum_{Y \in \mathcal{Y}_i : e \in E(Y)} w(i, e)x_{i,Y} + s(e, i)y_{e,i}\right) \\
&\quad = \frac{1}{t}\sum_{p=1}^{t} \sum_{i \in \{1,\ldots,k\} : e \in Y_i^{(p)}} \left(w(i, e) + \delta_e^{(p,i)} s(e, i)\right) \\
&\quad = \frac{u(e)}{\epsilon t} \ln\left(\prod_{p=1}^{t} \prod_{i \in \{1,\ldots,k\} : e \in Y_i^{(p)}} e^{\epsilon\frac{w(i,e) + \delta_e^{(p,i)} s(e,i)}{u(e)}}\right) \\
&\quad = \frac{u(e)}{\epsilon t} \ln\left(u(e)\omega_e^{(t)}\right) \\
&\quad \leq \frac{u(e)}{\epsilon t} \ln \mathcal{U}
\end{aligned} \tag{19}$$

By (18) and (19), we can choose $\lambda = \lambda_{ub} := \frac{1}{\epsilon t} \ln \mathcal{U}$ and have a feasible solution of (3). Comparing this upper bound with the lower bound (17), we can bound the approximation ratio ρ by

$$\rho \leq \frac{\lambda_{ub}}{\lambda_{lb}} \leq \frac{\epsilon' \ln \mathcal{U}}{\epsilon(1-\epsilon'\lambda^*) \ln\left(\frac{\mathcal{U}}{m+|\mathcal{M}|}\right)}. \tag{20}$$

As $\mathcal{U} = e^{\epsilon t \lambda_{ub}} \geq e^{\epsilon t \lambda^*} = (m+|\mathcal{M}|)^\alpha$ with $\alpha = \frac{\epsilon t \lambda^*}{\log(m+|\mathcal{M}|)}$ we have $\frac{\ln \mathcal{U}}{\ln\left(\frac{\mathcal{U}}{m+|\mathcal{M}|}\right)} \leq \frac{\alpha}{\alpha-1}$ and thus $\rho \leq \frac{\epsilon'}{\epsilon(1-\epsilon'\lambda^*)} \cdot \frac{\alpha}{\alpha-1}$. □

Together with binary search, this theorem implies a fully polynomial approximation scheme: Assuming $\frac{1}{2} \leq \lambda^* \leq 1$ and $1 \leq \Lambda \leq 2$, and given $0 < \epsilon_0 < 1$, we can choose ϵ_1 and ϵ_2 with $(1+\epsilon_1)(1+\epsilon_2) < \sqrt{1+\epsilon_0}$, $\epsilon := \frac{\epsilon_0}{27}$, and $t := \frac{\log(m+|\mathcal{M}|)}{\epsilon^2 \lambda^*}$ to obtain an approximation ratio of

$$\begin{aligned} \rho &= \tfrac{\epsilon'}{\epsilon(1-\epsilon'\lambda^*)(1-\epsilon)} \leq \sqrt{1+\epsilon_0}(1+\tfrac{3}{4}\Lambda\epsilon)\left(\tfrac{1}{1-\epsilon'}\right)^2 \leq \sqrt{1+\epsilon_0}\left(\tfrac{1}{1-\frac{3}{2}\epsilon}\right)^3 \\ &\leq \sqrt{1+\epsilon_0}\left(\tfrac{1}{1-\frac{1}{18}\epsilon_0}\right)^3 \leq \sqrt{1+\epsilon_0}\left(1+\tfrac{\epsilon_0}{9}\right)^3 \leq 1+\epsilon_0 \end{aligned} \tag{21}$$

in at most $\frac{1458 \ln(m+|\mathcal{M}|)}{\epsilon_0^2}$ iterations. Preliminary experience with practical instances, however, show much faster convergence and, with larger ϵ, good approximation guarantees already after 10–20 iterations.

Note that the Fractional Global Routing Algorithm also obtains an approximate dual solution, giving a certificate of infeasibility for most infeasible instances of the Global Routing Problem. This is true because the integrality gap is quite small, as will be shown in the next section.

We close this section by remarking that the above approximation algorithm is quite similar to the oldest and simplest global routing strategy, often called rip-up and reroute. The idea is to relax the capacity constraints, route each net by a shortest Steiner tree and then reroute nets using overloaded edges. This can be regarded as a Lagrangian relaxation method. The main differences to the above algorithm are the update of the dual variables (= Lagrange multipliers) and the missing randomized rounding step: traditional rip-up and reroute algorithms just output the last Steiner tree found for each net. Rip-up and reroute algorithms are useful for very easy instances, and also to post-optimize a solution, e.g. to remove local capacity violations introduced by randomized rounding.

4 Randomized Rounding

In this section we show how to round a fractional solution to (3) without increasing λ too much. We need the following lemma by Raghavan and Spencer (see Raghavan [1988]), a variation of Chernoff's [1952] bound.

Lemma 3. *Let $X_1, \ldots, X_k$ be independent random variables in $[0,1]$. Let μ be the sum of their expectations, and let $\epsilon > 0$. Then $X_1 + \cdots + X_k \leq (1+\epsilon)\mu$ with probability at least $1 - e^{\mu(\epsilon-(1+\epsilon)\ln(1+\epsilon))}$.*

Proof: Let Prob[$\cdot$] denote the probability of an event, and let Exp[$\cdot$] denote the expectation of a random variable. Using $(1+\epsilon)^x \le 1+\epsilon x$ for $0 \le x \le 1$ and $1+x \le e^x$ for $x \ge 0$ we compute $\mathrm{Prob}[X_1 + \cdots + X_k \ge (1+\epsilon)\mu] = \mathrm{Prob}\left[\frac{\prod_{i=1}^k (1+\epsilon)^{X_i}}{(1+\epsilon)^{(1+\epsilon)\mu}} \ge 1\right] = \mathrm{Prob}\left[\frac{\prod_{i=1}^k (1+\epsilon X_i)}{(1+\epsilon)^{(1+\epsilon)\mu}} \ge 1\right] \le \mathrm{Exp}\left[\frac{\prod_{i=1}^k (1+\epsilon X_i)}{(1+\epsilon)^{(1+\epsilon)\mu}}\right] = \frac{\prod_{i=1}^k (1+\epsilon\,\mathrm{Exp}[X_i])}{(1+\epsilon)^{(1+\epsilon)\mu}} \le \frac{\prod_{i=1}^k e^{\epsilon\,\mathrm{Exp}[X_i]}}{(1+\epsilon)^{(1+\epsilon)\mu}} = \frac{e^{\epsilon\mu}}{(1+\epsilon)^{(1+\epsilon)\mu}} = e^{(\epsilon-(1+\epsilon)\ln(1+\epsilon))\mu}$. □

Similarly as Raghavan [1988], we can now bound the effect of randomized rounding:

Theorem 4. *Given a fractional solution (x, y, λ) to (3), we obtain a rounded solution $(\hat{x}, \hat{y}, \hat{\lambda})$ as follows: Independently for all $i \in \mathcal{N}$ we choose $Y \in \mathcal{Y}_i$ as Y_i with probability $x_{i,Y}$; then we set $\hat{x}_{i,Y_i} := 1$ and $\hat{x}_{i,Y} := 0$ for all $Y \in \mathcal{Y}_i \setminus \{Y_i\}$. Next we set $\hat{y}_{i,e} := \frac{y_{i,e}}{\sum_{Y\in\mathcal{Y}_i : e\in E(Y)} x_{i,Y}}$ if $e \in E(Y_i)$ and $\hat{y}_{i,e} := 0$ otherwise. Finally, we choose $\hat{\lambda}$ minimum possible such that $(\hat{x}, \hat{y}, \hat{\lambda})$ is a feasible solution to (3).*

Let $\Lambda \le \frac{U(M)}{c(M,i)\sum_{e\in E(Y)} l(e,i)}$ for $i \in M \in \mathcal{M}$ and $Y \in \mathcal{Y}_i$, and $\Lambda \le \frac{u(e)}{w_{i,e}+s(e,i)}$ for $i \in \mathcal{N}$ and $e \in E(G)$.

Then $\hat{\lambda} \le e^{\alpha}\lambda$ with probability at least $\frac{1}{2}$, where α is the solution to $1 + (\alpha - 1)e^{\alpha} = \frac{\ln(2(|\mathcal{M}|+|E(G)|))}{\lambda\Lambda}$.

Proof: We write $y'_{i,e} := \frac{y_{i,e}}{\sum_{Y\in\mathcal{Y}_i : e\in E(Y)} x_{i,Y}}$; then $\hat{y}_{i,e} = \sum_{Y\in\mathcal{Y}_i : e\in E(Y)} y'_{i,e}\hat{x}_{i,Y}$. $\hat{\lambda}$ is constrained by $|\mathcal{M}| + |E(G)|$ linear inequalities, which can be written as $A\hat{x} \le \hat{\lambda}\mathbb{1}$, where $\mathbb{1}$ is an all-one vector, and the entries of the matrix A are $A_{M;i,Y} = \frac{c(M,i)}{U(M)}\sum_{e\in E(Y)}(l(e,i) - v(e,i)y'_{i,e})$ (for $i \in M \in \mathcal{M}$ and $Y \in \mathcal{Y}_i$) and $A_{e;i,Y} = \frac{1}{u(e)}(w(i,e) + s(e,i)y'_{i,e})$ (for $i \in \mathcal{N}$, $Y \in \mathcal{Y}_i$ and $e \in E(Y)$).

Note that

$$\Lambda\frac{c(M,i)}{U(M)}\sum_{Y\in\mathcal{Y}_i}\sum_{e\in E(Y)}(l(e,i) - v(e,i)y'_{i,e})\hat{x}_{i,Y}, \tag{22}$$

$i = 1, \ldots, k$, are independent random variables in $[0, 1]$. Similarly,

$$\Lambda\frac{1}{u(e)}\left(w(i,e) + s(e,i)y'_{i,e}\right)\sum_{Y\in\mathcal{Y}_i : e\in Y}\hat{x}_{i,Y}, \tag{23}$$

$i = 1, \ldots, k$, are independent random variables in $[0, 1]$.

In each case, the sum of the expectations of the k random variables is at most $\lambda\Lambda$. Hence, by Lemma 3, their sum is at least $(1+\epsilon)\lambda\Lambda$ with probability at most $e^{\lambda\Lambda(\epsilon-(1+\epsilon)\ln(1+\epsilon))}$. Setting $\epsilon = e^{\alpha} - 1$, this is $e^{-\lambda\Lambda(1+(\alpha-1)e^{\alpha})} = \frac{1}{2(|\mathcal{M}|+|E(G)|)}$. □

The algorithm can easily be derandomized by applying the method of conditional probabilities in a standard way (Raghavan [1988]), however, this may be too slow in practice. By allowing different values of Λ for the constraints, the above bound could be improved. Another attempt of improving the bound has been made by Leighton et al. [2001], who showed how the (de)randomized rounding can be improved by using an algorithmic version of the Lovász Local Lemma, exploiting the fact that most Steiner trees are short and their rounding does not affect regions that are far away.

In practice, violations introduced by randomized rounding are much smaller than guaranteed by the above theorem, and they can usually be removed by simple postprocessing heuristics.

5 Discussion

Note that the global routing problem could also be used to model optimizing traffic flows in a static setting (flow continuously repeated over time, like in rush-hour traffic), with hard capacities on edge flow rate and with transit times that depend piecewise linearly on the congestion. The algorithm is actually equivalent to selfish routing, but with respect to edge costs depending exponentially, rather than linearly, on the congestion. It is known that general selfish routing can lead to arbitrarily bad solutions (called user equilibria or Nash equilibria; cf. Roughgarden and Tardos [2000], Schulz and Stier [2003]). However, it is not clear whether this also holds for *fair* selfish routing, where the users (commodities, nets) take turns in finding alternate shortest routes (like in our algorithm). This may lead to better user equilibria. It is an interesting question whether there is a class of (reasonable) latency functions where fair selfish routing converges to the system optimum. Moreover, it is an open question whether the algorithm can be extended to a model in which coupling capacitances (travel times) depend non-linearly on the congestion.

Algorithms for packing linear programs, and more generally (nonlinear) min-max resource sharing problems, have been proposed before by many authors, including Grigoriadis and Khachiyan [1994,1996], Plotkin, Shmoys and Tardos [1995], Young [1995], and Garg and Könemann [1998]. However, the number of iterations in all these approaches depends at least linearly on the number of blocks (nets in our context) or on the number of resource constraints (edges plus capacitance constraints) or on the width (the value of the worst possible solution). In our algorithm the number of iterations is independent of the number of blocks and the width, and depends only logarithmically on the number of resource constraints. The dependency on the performance guarantee in all these algorithms is $O(\epsilon^{-2})$, like in our algorithm, or worse. It may be interesting to try to extend our approach to more general contexts.

With Dirk Müller, we are currently incorporating the presented algorithm into *BonnRoute*, an automatic routing tool used for designing many of the most complex industrial chips. We are confident that this will improve the quality of the chips (speed, power consumption), increase the manufacturing yield, and shorten turn-around times in design closure by avoiding semi-manual postprocessing.

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