Exercises 2

Exercise 1:
Let $\alpha(G)$ denote the size of a maximum stable set in a graph $G$, and $\zeta(G)$ the minimum cardinality of an edge cover.
(An edge cover in $G$ is a set $F \subseteq E(G)$ of edges such that every vertex of $G$ is incident to at least one edge in $F$.)
Prove:
(a) $\alpha(G) + \tau(G) = |V(G)|$ for any graph $G$.
(b) $\nu(G) + \zeta(G) = |V(G)|$ for any graph $G$ with no isolated vertices.
(c) $\zeta(G) = \alpha(G)$ for any bipartite graph $G$ with no isolated vertices.

(1 + 2 + 1 points)

Exercise 2:
(a) Let $S = \{1, 2, \ldots, n\}$ and $0 \leq k < \frac{n}{2}$. Let $A$ and $B$ be the collection of all $k$-element and $(k + 1)$-element subsets of $S$, respectively. Construct a bipartite graph

$$G = (A \cup B, \{\{a, b\} : a \in A, b \in B, a \subseteq b\}).$$

Prove that $G$ has a matching covering $A$.

(b) Prove Sperner’s Lemma: the maximum number of subsets of an $n$-element set such that none is contained in any other is $\left(\binom{n}{\left\lfloor \frac{n}{2}\right\rfloor}\right)$.

(2 + 2 points)

Exercise 3:
Let $G$ be a bipartite graph with bipartition $V(G) = A \cup B$. Suppose that $S \subseteq A$, $T \subseteq B$, and there is a matching covering $S$ and a matching covering $T$. Prove that then there is a matching covering $S \cup T$.

(4 points)

Exercise 4:
Prove that every 3-regular simple graph with at most two bridges has a perfect matching. Is there a 3-regular simple graph without a perfect matching?

$\textbf{Hint:}$ Use Tutte’s Theorem 10.13.

(4 points)

Deadline: Tuesday, October 26th, before the lecture.