Exercise Set 8

Exercise 8.1. For an undirected graph G, let P_G denote the spanning-tree polytope of G and

$$Q_G := \left\{ x \in [0,1]^{E(G)} : \sum_{e \in E(G)} x_e = |V(G)| - 1 \ , \ \sum_{e \in \delta(X)} x_e \geq 1 \text{ for } \emptyset \neq X \subsetneq V(G) \right\}.$$

Prove:

(i) $P_G \subseteq Q_G$ for every graph G.

(ii) There exists a graph G with $P_G \neq Q_G$.

(1+2 points)

Exercise 8.2. Let G be an undirected graph. Given a partition $(X_1, \ldots, X_k)$ of $V(G)$ we define $\delta(X_1, \ldots, X_k) := \delta(X_1) \cup \cdots \cup \delta(X_k)$ (so, in particular, if $\emptyset \neq X \subsetneq V(G)$ we have $\delta(X) = \delta(X, V(G) \setminus X)$). Consider the polytope

$$R_G := \left\{ x : E(G) \to [0,1] \ : \ \sum_{e \in E(G)} x(e) = |V(G)| - 1 \text{ and } \sum_{e \in \delta(X_1, \ldots, X_k)} x(e) \geq k - 1 \text{ for every partition } (X_1, \ldots, X_k) \text{ of } V(G) \right\}$$

Show that R_G is the spanning-tree polytope of G.

(5 points)

Exercise 8.3. Let G be a 2-edge-connected graph, and let $T := \{v \in V(G) : |\delta(v)| \text{ odd}\}$.

(i) Prove that $x \in \mathbb{R}^{E(G)}$ with $x_e = \frac{1}{3}$ for all $e \in E(G)$ is a convex combination of incidence vectors of T-joins.

(ii) Show that a connected Eulerian subgraph H of $2G$ with $V(H) = V(G)$ and $|E(H)| \leq \frac{2}{3}\Big(|V(G)| + |E(G)| - 1\Big)$ can be computed in polynomial time.

(3+3 points)

Exercise 8.4. Let G be a graph. A *2-cover* of G is a function $y : V(G) \to \{0, 1, 2\}$ with $y(v) + y(w) \geq 2$ for all $\{v, w\} \in E(G)$. The *size* of y is $\sum_{v \in V(G)} y(v)$.

If y is a 2-cover, the set $\{v \in V(G) \ : \ y(v) = 0\}$ is a stable set.
Conversely, a stable set $A \subseteq V(G)$ determines a 2-cover y by setting

$$y(v) = \begin{cases} 0 & \text{if } v \in A, \\ 2 & \text{if } v \in \Gamma(A), \\ 1 & \text{otherwise.} \end{cases}$$

Prove:

(i) The maximum size of a 2-matching in G equals the minimum size of a 2-cover of G, where the *size* of a 2-matching $f : E(G) \to \{0, 1, 2\}$ is $\sum_{e \in E(G)} f(e)$.

(ii) G has a perfect 2-matching iff $|\Gamma(A)| \geq |A|$ for all stable sets $A \subseteq V(G)$.

(3+3 points)

Deadline: December 5th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws19/co_exercises/exercises.html`

In case of any questions feel free to contact me at rabenstein@or.uni-bonn.de.