Professor Dr. Stephan Held
Stefan Rabenstein

Exercise Set 11

Exercise 11.1. Prove that a nonempty compact set $P \subseteq \mathbb{R}^n_+$ is a polymatroid if and only if

(i) For all $0 \le x \le y \in P$ we have $x \in P$.

(ii) For all $x \in \mathbb{R}^n_+$ and all $y, z \le x$ with $y, z \in P$ that are maximal with this property (i.e. $y \le w \le x$ and $w \in P$ implies $w = y$, and $z \le w \le x$ and $w \in P$ implies $w = z$) we have $\mathbb{1}y = \mathbb{1}z$, where $\mathbb{1}$ is the vector whose entries are all 1.

(4 points)

Exercise 11.2. Let (G, u, s, t) be a network and $U := \delta^+(s)$. Let

$$P := \left\{x \in \mathbb{R}^U_+ : \text{there is an } s\text{-}t \text{ flow } f \text{ in } (G, u) \text{ with } f(e) = x_e \text{ for all } e \in U\right\}.$$

Prove that P is a polymatroid.

(4 points)

Exercise 11.3. Let $f : 2^U \to \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$. Prove that the set of vertices of the base polyhedron of f is precisely the set of vectors $b^{\prec}$ for all total orders $\prec$ of U, where

$$b^{\prec}(u) := f\Big(\{v \in U : v \preceq u\}\Big) - f\Big(\{v \in U : v \prec u\}\Big) \qquad (u \in U).$$

(6 points)

Exercise 11.4. Let $f : 2^U \to \mathbb{R}$ be a submodular function with $f(\emptyset) = 0$, and let $B(f)$ denote its base polyhedron. Prove that

$$\begin{aligned}
&\min\{f(X) : X \subseteq U\} \\
&= \max\left\{\sum_{u \in U} z_u : z \in \mathbb{R}^U \text{ with} \sum_{u \in A} z_u \le \min\{0, f(A)\} \text{ for all } A \subseteq U\right\} \\
&= \max\left\{\sum_{u \in U} \min\{0, y_u\} : y \in B(f)\right\}.
\end{aligned}$$

(6 points)

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Deadline: January 9$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws19/co_exercises/exercises.html`

In case of any questions feel free to contact me at rabenstein@or.uni-bonn.de.