Winter term 2021/22
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Combinatorics, Graphs, Matroids
Assignment Sheet 2

1. Let E be a finite set and $\mathcal{B} \subseteq 2^E$. Show that $\mathcal{B}$ is the set of bases of a matroid if and only if the following three conidtions are satisfied:

   (B1) $\mathcal{B} \neq \emptyset$

   (B2)' For $B_1, B_2 \in \mathcal{B}$ and $x \in B_1$, there is an element $y \in B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

   (B3) For $B_1, B_2 \in \mathcal{B}$, we have: $B_1 \subseteq B_2 \Rightarrow B_1 = B_2$. (3 points)

2. Let $(E, \mathcal{F})$ be a matroid with rank function r, and let k be a positive integer. Let $r_k : 2^E \to \mathbb{Z}_+$ be defined by $r_k(X) = \min\{k, r(X)\}$. Show that r_k is the rank function of a matroid. (3 points)

3. Let $(E, \mathcal{F})$ be a matroid with closure operator $\sigma : 2^E \to 2^E$. Let $X, Y \subseteq E$ be sets with $\sigma(X) = X$ and $\sigma(Y) = Y$. Prove or disprove the following statements:

   (a) These conditions imply $\sigma(X \cap Y) = X \cap Y$.

   (b) These conditions imply $\sigma(X \cup Y) = X \cup Y$. (2+2 points)

4. Let $(E, \mathcal{F})$ be a matroid, and let $\mathcal{C}$ be the set of circuits of $(E, \mathcal{F})$. Moreover, let $x \in E$. Consider the following sets $\mathcal{C}_i$. Either give a proof that, under theses conditions, C_i is the set of circuits of a matroid or show by an example that this is not necessarily the case.

   (a) $\mathcal{C}_1 = \{C \in 2^E \mid C \in \mathcal{C} \text{ and } x \notin C\}$

   (b) $\mathcal{C}_2 = \{C \in 2^E \mid (C \cup \{x\}) \in \mathcal{C}\}$

   (c) $\mathcal{C}_3 = \{C \in 2^E \mid x \in C \text{ and } (C \setminus \{x\}) \in \mathcal{C}\}$ (2+2+2 points)

Du date: Thursday, October 28, 2021, before the lecture (in the lecture hall)