Winter term 2021/22
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Combinatorics, Graphs, Matroids
Assignment Sheet 3

1. Let $(E, \mathcal{F})$ be a matroid, and let A and B be two subsets of E, each containing a basis, with $|A| > |B|$. Must there necessarily be an $x \in A \setminus B$ such that $A \setminus \{x\}$ contains a basis? Prove the correctness of your answer. (4 points)

2. Let $(E, \mathcal{F})$ be a matroid and $c : E \to \mathbb{R}$ a mapping with $c(e) \neq c(e')$ and $c(e) \neq 0$ for all $e, e' \in E$ with $e \neq e'$. Show that both the maximization problem and the minimization problem for $(E, \mathcal{F})$ have a unique solution. Show by example that this is not necessarily the case if $(E, \mathcal{F})$ is only an independence system. (3 points)

3. Let $w_1, \dots, w_n$ and W be positive integers, and let the independence system $(E, \mathcal{F})$ be given by $E = \{1, \dots, n\}$ and

$$\mathcal{F} = \{F \subseteq E \mid \sum_{j \in F} w_j \leq W\}.$$

Give the smallest possible rank quotient for $(E, \mathcal{F})$. Prove the correctness of your answer. (3 points)

4. Show that for matroids the independence and basis-superset oracles are equivalent. (2 points)

5. Let k be a positive integer. For a graph G let

$$\mathcal{F}_G = \{F \subseteq E(G) \mid \Delta((V(G), F)) \leq k\}.$$

   (a) Show that $(E(G), \mathcal{F}_G)$ is always an independence system but not necessarily a matroid.

   (b) Consider the problem to find, given a graph G with edge weights $c : E(G) \to \mathbb{R}_+$, a set $F \in \mathcal{F}_G$ maximizing $\sum_{e \in F} c(e)$. Show that the Best-In-Greedy finds a solution for this optimization problem that is at most by a factor of 2 worse than an optimum solution. (2+2 points)

Remark: $\Delta((V, E))$ denotes the maximum vertex degree of the graph (V, E).

Du date: Thursday, November 4, 2021, before the lecture (in the lecture hall)