Combinatorics, Graphs, Matroids
Assignment Sheet 5

Preliminary remark: In this exercise sheet, you can use the statement that an undirected graph is planar if and only if it contains neither the K_5 nor the $K_{3,3}$ as a minor, even if the proof of this statement has not been completed in the lecture.

1. Let G be a directed graph such that for all $v, w \in V(G)$ exactly on v-w-path exists in G. Show that G Eulerian. (3 points)

2. What is the smallest n such that there exists a nonplanar simple graph G with n nodes whose complement $\bar{G}$ is also nonplanar? The graph $\bar{G}$ is defined by $V(\bar{G}) = V(G)$ and by the property that two nodes in $\bar{G}$ are connected by an edge if and only if they are not connected in G. (4 points)

3. Given a graph G and an edge $e = \{v, w\} \in E(G)$ we say that H results from G by subdividing e if $V(H) = V(G) \dot{\cup} \{x\}$ and $E(H) = (E(G) \setminus \{e\}) \cup \{\{v, x\}, \{x, w\}\}$. A graph resulting from G by successively subdividing edges is called a subdivision of G. Show the following statements:

   (a) If H contains a subdivision of G then G is a minor of H. The converse is not true.

   (b) If a graph contain a $K_{3,3}$ or K_5 minor then it also contains a subdivision of $K_{3,3}$ or of K_5. (2+3 points)

   Remark: Together the two statements show that a graph is planar if and only if no subgraph is a subdivision of $K_{3,3}$ or K_5.

4. Show that the following statements are equivalent:

   (a) For every infinite sequence $G_1, G_2 \ldots$ of graphs, there are two indices $i < j$ such that G_i is a minor of G_j.

   (b) Let $\mathcal{G}$ be a class of graphs that is closed under computing minors i.e. for each $G \in \mathcal{G}$ every minor of G is contained in $\mathcal{G}$, too. Then there is a finite set $\mathcal{X}$ of graphs such that $\mathcal{G}$ consist of exactly the graphs not containing an element of $\mathcal{X}$ as a minor. (4 points)

   Remark: The statements are not only equivalent but also true. Choose for example the set of planar graphs as $\mathcal{G}$. This set is closed under computing minors and the set $\mathcal{X}$ can consist of $K_{3,3}$ and K_5.

Du date: Thursday, November 18, 2021, before the lecture (in the lecture hall)