Winter term 2021/22
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Combinatorics, Graphs, Matroids
Assignment Sheet 7

1. For a simple undirected graph G let $t(G)$ be the smallest number such that there are planar graphs $G_1, \dots, G_{t(G)}$ satisfying the following conditions:

   - $V(G_i) = V(G)$ $(i \in \{1, \dots, t(G)\})$,
   - $E(G) = \bigcup_{i=1}^{t(G)} E(G_i)$.

   This means that a graph G is planar if and only if $t(G) = 1$.

   (a) Show that $t(K_n) \geq \lfloor \frac{n+7}{6} \rfloor$ holds for $n \in \mathbb{N} \setminus \{0\}$.

   (b) Show that there is a graph G with $t(G) = 2$ and $\chi(G) = 8$.

   (c) Give as good an upper bound as possible for the following value:

   $$\max\{\chi(G) \mid t(G) = 2\}.$$

   Prove the correctness of your bound. (2+2+2 points)

2. For an undirected graph G and $t \in \mathbb{N}$ let $p_G(t)$ be the number of different feasible node colorings of G with the colors $\{1, \dots, t\}$. Here we consider two node colorings to be different if there is at least one node to which they assign different colors. Show that for each graph G the mapping $p_G : \mathbb{N} \to \mathbb{N}$ is a polynomial of degree $n = |V(G)|$. What is the coefficient of t^n and what is the coefficient of t^{n-1}? (4 points)

   Hint: First consider complete graphs. For graphs in which there are two nodes v and w that are not connected by an edge, you can then consider what happens if you connect v and w by an additional edge or contract $\{v, w\}$.

3. Show (using the Four-Color Theorem) that the edge chromatic number of a 3-regular planar graph G with no bridges (i.e., no edges whose deletion would increase the number of components of G) is 3. Does this statement still hold if instead of 3-regularity we require only $\Delta(G) \leq 3$? (4 points)

4. Let G be a regular graph with odd number of nodes, and let H arise from G by deleting at most $\frac{\Delta(G)}{2} - 1$ edges (in particular $G = H$ is possible). Show that this implies $\chi'(H) = \Delta(H) + 1$. (2 points)

Du date: Thursday, December 2, 2021, before the lecture (in the lecture hall)